ON THREE CONJECTURES ABOUT THE SECANT DEFECTIVITY OF CLASSICALLY STUDIED VARIETIES

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1. Introduction

This paper and the subsequent paper that appears in the same volume are based on the two talks the author gave at the algebraic geometry symposium held at Waseda University in Tokyo, Japan, from November 10 to November 13, 2010. The titles of these two talks were "On the secant defectivity of Segre-Veronese varieties I & II", both of which explored problems related to secant varieties of not only Segre-Veronese varieties, but also other classically studied varieties such as Segre varieties, Veronese varieties, and Grassmann varieties.

This paper summarizes the first of the two talks. The goal of the first talk was twofold; namely to provide a brief introduction to the field of secant varieties and to discuss recent progress towards the completion of three conjectures on the classification of defective secant varieties (i.e., secant varieties, which do no have the expected dimension) of classically known varieties. The author thus changed the title to "On three conjectures about the secant defectivity of classically studied varieties" to more reflect the contents of the talk.

The main focus of the second talk was on the more detailed study of higher secant varieties of Segre-Veronese varieties. The second paper will therefore restrict its attention to recent joint work with Chiara Maria Brambilla on the classification of defective secant varieties for the Segre-Veronese case.

2. Waring's problem for polynomials and secant varieties of Veronese varieties

2.1. **Polynomial Waring's problem.** In 1770, Edward Waring suggested the problem of finding a positive integer s_0 such that every positive integer can be written as the sum of at most s_0 d^{th} powers of positive integers. This problem was solved affirmatively by Hilbert in 1909. There are several variations of this problem for polynomials known. Here we mean by *Waring's problem for polynomials* the following:

Problem 2.1. *What is the smallest positive integer s_0 such that a general homogeneous polynomial of degree d in $(n+1)$ variables is expressible as the sum of s d^{th} powers of linear forms?*

In the preceding two paragraphs, we will see that this problem can be very naturally translated into a classical problem in algebraic geometry.

2.2. **Secant varieties.** Let K denote an algebraically closed field of characteristic 0 and let V be an $(n+1)$-dimensional vector space over K. Throughout this paper, we denote by

- $V^* = \mathrm{Hom}_K(V, K)$ the dual of V,
- $\mathrm{Sym}_d V$ the d^{th} symmetric power of V,
- $\mathbb{P}V$ the projective space of V,
- $[v] \in \mathbb{P}V$ the equivalence class containing $v \in V \setminus \{0\}$,
- $\langle S \rangle$ the linear span of a subset S of $\mathbb{P}V$,
- $\widehat{X}$ the affine cone over a variety $X \subset \mathbb{P}V$ in V.
- $\mathbb{T}_p(X)$ the projective tangent space to a variety $X \subset \mathbb{P}V$ at a $p \in X$.

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For a given positive integer s, let $p_1, \dots, p_s$ be linearly independent points of X. Then $\langle p_1, \dots, p_s \rangle$ is called a *secant* $(s-1)$*-plane* to X. The Zariski closure of the union of all secant $(s-1)$-planes to X is called the s^{th} *secant variety* of X and denoted by $\sigma_s(X)$.

2.3. **Geometric interpretation of Waring's problem.** Consider the d^{th} Veronese map

$$v_d : \mathbb{P}V^* \to \mathbb{P}\mathrm{Sym}_d V^*,$$

i.e., the map given by $[\ell] \mapsto [\ell^d]$. The s^{th} secant variety $\sigma_s(v_d(\mathbb{P}V))$ of the Veronese variety can be viewed as the Zariski closure of the set of projectivizations of homogeneous polynomials of degree d in $(n+1)$ variables that are expressible as the sum of s d^{th} powers of linear forms. Therefore the polynomial Waring problem can be reformulated as follows:

Problem 2.2. *What is the smallest positive integer s_0 such that $\sigma_{s_0}(v_d(\mathbb{P}V)) = \mathbb{P}\mathrm{Sym}_d V$?*

It goes out without saying that $\sigma_{s_0}(v_d(\mathbb{P}V)) = \mathbb{P}\mathrm{Sym}_d V$ if and only if

$$\dim \sigma_{s_0}(v_d(\mathbb{P}V)) = \dim \mathbb{P}\mathrm{Sym}_d V = \binom{n+d}{d}.$$

Thus the main task of answering the above-mentioned question is to find $\dim \sigma_s(v_d(\mathbb{P}V))$ for each positive integer s and determine for which s the dimension of $\sigma_s(v_d(\mathbb{P}V))$ achieves $\binom{n+d}{d}$. In the next paragraph, we will discuss the "expected dimension" of the secant variety of a projective variety and present a celebrated theorem of Alexander and Hirschowitz, which states that every secant variety of a Veronese variety has the expected dimension modulo a fully described list of exceptions.

2.4. **Defectivity of secant varieties.** Let $X \subseteq \mathbb{P}V$ be a projective variety. A naive dimension count implies

$$\dim \sigma_s(X) \leq \min \{ s(\dim X + 1) - 1, N \}. \tag{2.1}$$

If equality holds in (2.1), we say that $\sigma_s(X)$ has the *expected dimension.* $\sigma_s(X)$ is said to be *defective* if it does not have the expected dimension. We say that X is *defective* if $\sigma_s(X)$ is defective for some s. A natural question to ask is whether every secant variety is non-defective.

Example 2.3. Let V be a three-dimensional vector space over $\mathbb{C}$. Then $\sigma_2(v_2(\mathbb{P}V^*))$ is expected to fill $\mathbb{P}\mathrm{Sym}_2 V^* = \mathbb{P}^5$, because

$$\min\{2 \cdot (2+1) - 1, 5\} = 5.$$

Note that $\mathbb{P}\mathrm{Sym}_2 V^*$ is the projectivization of the space of non-zero 3×3 symmetric matrices with entries from $\mathbb{C}$. Thus $\sigma_s(v_2(\mathbb{P}V^*))$ can be viewed as the space of non-zero 3×3 symmetric matrices of rank at most s. Therefore one can immediately see that $\sigma_1(v_2(\mathbb{P}V^*)) = v_2(\mathbb{P}V^*)$ may be expressed as the common zero locus of the 2×2 minors of the generic symmetric matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_3 & z_4 \\ z_2 & z_4 & z_5 \end{pmatrix},$$

$\sigma_2(v_2(\mathbb{P}V^*))$ is contained in the cubic hypersurface given by the determinant of the generic symmetric matrix, and $\sigma_3(v_2(\mathbb{P}V^*)) = \mathbb{P}^5$. In particular, $\sigma_2(v_2(\mathbb{P}V^*))$ is defective, because $\dim \sigma_2(v_2(\mathbb{P}V^*)) \leq 4 < 5$.

Note that the same argument shows that $\sigma_s(v_2(\mathbb{P}V))$ is defective if and only if $n \geq 2$ and $2 \leq s \leq n$. This means that if $d = 2$, then $v_d(\mathbb{P}V)$ is defective for almost all n. This trend does not however continues as d gets greater.

Theorem 2.4 ([5])**.** *Let $d \geq 3$. Then $\sigma_s(v_d(\mathbb{P}V))$ is defective if and only if*

(i) *$d = 3$, $n = 4$ and $s = 7$ or*
(ii) *$d = 4$ and $(n, s) = (2, 5)$, (3.9) or $(4, 14)$.*

Note that $\sigma_s(v_d(\mathbb{P}V))$ is expected to fill $\mathbb{P}\mathrm{Sym}_d V$ for every $s \cdot (n+1) \geq \binom{n+d}{d}$. The Alexander-Hirschowitz theorem therefore completes the Waring problem for the polynomial.

Corollary 2.5. *A general homogeneous polynomial of degree d in $(n+1)$ variables is expressible as the sum of*

$$s_0(n,d) = \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$$

d^{th} *powers of linear forms except for $s_0(4,3) = 8$, $s_0(2,4) = 6$, $s_0(3,4) = 10$, $s_0(4,4) = 15$ and $s_0(n,2) = n+1$.*

3. Secant varieties of other classically studied varieties

As was shown in Subsection 2.4, the classification of defective secant varieties of Veronese varieties has been completed. In this section, we provide corresponding conjecturally complete lists of defective secant varieties for Segre varieties, for Grassmann varieties and for two-factor Segre-Veronese varieties.

3.1. **Segre varieties.** For each $i \in \{1, \ldots, k\}$ with $k \geq 2$, let $n_i \in \mathbb{N}$ with $n_1 \leq \cdots \leq n_k$ and let V_i be an (n_i+1)-dimensional vector space over K. The image of the Segre map, i.e., the map

$$\mathrm{Seg} : \prod_{i=1}^{k} \mathbb{P}V_i \to \mathbb{P}\left(\bigotimes_{i=1}^{k} V_i\right)$$

given by sending $([v_1], \ldots, [v_k])$ to $[v_1 \otimes \cdots \otimes v_k]$, is called a *Segre variety.*

Definition 3.1. A k-tuple $\mathbf{n} = (n_1, \ldots, n_k)$ of positive integers is said to be *balanced* if

$$n_k \leq \prod_{i=1}^{k-1}(n_i+1) - \sum_{i=1}^{k-1} n_i.$$

Otherwise $\mathbf{n}$ is said to be *unbalanced.*

Conjecture 3.2 ([3])**.** $\sigma_s\left(\mathrm{Seg}\left(\prod_{i=1}^{k} \mathbb{P}V_i\right)\right)$ *is defective if and only if*

(i) $\mathbf{n} = (n_1, n_2)$ *with* $n_1, n_2 \geq 2$ *and* $2 \leq s \leq n$.
(ii) $\mathbf{n}$ *is unbalanced and*

$$\prod_{i=1}^{k-1}(n_i+1) - \sum_{i=1}^{k-1} < s < \min\left\{\prod_{i=1}^{k-1}(n_i+1),\ n_k\right\}.$$

(iii) $\mathbf{n} = (2, n, n)$ *with n even and* $s = (3n+2)/2$.
(iv) $\mathbf{n} = (2, 3, 3)$ *and* $s = 5$, *or*
(v) $\mathbf{n} = (1, 1, n, n)$ *and* $s = 2n+1$.

3.2. **Grassmann varieties.** Let V be an $(n+1)$-dimensional vector space over K and let $\mathbb{G}(k,n) \subset \mathbb{P}\left(\bigwedge^{k+1} V\right)$ be the Plücker embedding of k-planes of V.

Conjecture 3.3 ([17], [6])**.** $\sigma_s(\mathbb{G}(k,n))$ *has the expected dimension, i.e.,*

$$\dim \sigma_s(\mathbb{G}(k,n)) = \min\left\{s((k+1)(n-k)+1) - 1,\ \binom{n+1}{k+1} - 1\right\},$$

except for

(i) $\sigma_s(\mathbb{G}(1,n))$ *with* $n \geq 3$ *and* $2 \leq s \leq n$;
(ii) $\sigma_3(\mathbb{G}(3,7))$;
(iii) $\sigma_4(\mathbb{G}(3,7))$, *and*
(iv) $\sigma_4(\mathbb{G}(2,8))$.

3.3. **Segre-Veronese varieties.** Let $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$ and let V_i be an $(n_i + 1)$-dimensional vector space over K for each $i \in \{1, \ldots, k\}$. Then $\mathrm{Seg}\left(\prod_{i=1}^k v_{d_i}(\mathbb{P}V_i)\right) \subset \mathbb{P}\left(\bigotimes_{i=1}^j \mathrm{Sym}_{d_i} V_i\right)$ is called a *Segre-Veronese variety*. Unfortunately, higher secant varieties of Segre-Veronese varieties are less well-understood. We know of no general conjecture for defective secant varieties of such varieties. Recently considerable efforts have been made to develop techniques to study secant varieties of two-factor Segre-Veronese varieties and the following conjecture was suggested:

Conjecture 3.4 ([1]). *$\sigma_s\left(\mathrm{Seg}\left(\prod_{i=1}^2 v_{d_i}(\mathbb{P}V_i)\right)\right)$ is defective if and only if $(\mathbf{n}, \mathbf{d}, s)$ falls into one the following:*

$\mathbf{n}$	$\mathbf{d}$	s
(m, n) *with* $m \geq 2$	$(d, 1)$	$\binom{m+d}{d} - m < s < \min\left\{\binom{m+d}{d} n + 1\right\}$
$(2, 2k+1)$	$(1, 2)$	$3k + 2$
$(4, 3)$	$(1, 2)$	6
$(1, 2)$	$(1, 3)$	5
$(1, n)$	$(2, 2)$	$n + 2 \leq s \leq 2n + 1$
$(2, 2)$	$(2, 2)$	$7,\ 8$
$(2, n)$	$(2, 2)$	$\left\lfloor \frac{3n^2+9n+5}{n+3} \right\rfloor \leq s \leq 3n + 2$
$(3, 3)$	$(2, 2)$	$14,\ 15$
$(3, 4)$	$(2, 2)$	19
$(n, 1)$	$(2, 2k)$	$kn + k + 1 \leq s \leq kn + k + n$

Evidence for this conjecture was provided by results in [4, 7, 8, 9, 11, 10, 13]. Further evidence in support of the conjecture was obtained via the computational experiments using `Macaulay 2` [16].

4. Basic tools

One of the most fundamental questions centered around secant varieties is to find their dimensions. Note that, in order to find the dimension of a given variety, it is enough to calculate the dimension of its tangent space at a generic point. The first part of this section discusses a theorem called *Terracini's lemma*, which says that the linear subspace spanned by the tangent spaces of a projective variety at s generic points is the tangent space to the secant variety of the projective variety at a generic point of the linear span of the s generic points. The second part of this section will be devoted to describing an inductive approach for computing the dimension of the tangent space to a secant variety that is based on specialization of points on subvarieties and projection.

4.1. **Terracini's lemma.** Let $X \subseteq \mathbb{P}^N$ be a projective variety and let $I_X = (F_0, \ldots, F_\ell) \subseteq K[x_0, \ldots, x_N]$ be the ideal of X. Then one can compute the dimension of the affine cone $\widehat{\mathbb{T}}_p(X)$ over the projective tangent space $\mathbb{T}_p(X)$ to X at a generic point $p \in X$ as follows:

Step 1. Compute the Jacobian matrix $\mathrm{Jac}(I_X) = \frac{\partial(F_0, \ldots, F_\ell)}{\partial(x_0, \ldots, x_n)}$ of I_X.

Step 2. Evaluate $\mathrm{Jac}(I_X)$ at p of X.

Step 3. Find the rank of $\mathrm{Jac}(I_X)_p$.

Then $\dim \widehat{\mathbb{T}}_p(X) = N + 1 - \operatorname{rank} \mathrm{Jac}(I_X)_p$. Thus if we have $I_{\sigma_s(X)}$ and a generic point q of $\sigma_s(X)$, then we can compute $\dim \sigma_s(X)$ by computing $\widehat{\mathbb{T}}_p(X)$ as described above. Unfortunately, it is quite expensive to compute $I_{\sigma_s(X)}$ from I_X in general. The following theorem implies, however, that if I_X is given, then one can compute $\dim \sigma_s(X)$ without knowing the ideal of $\sigma_s(X)$:

Theorem 4.1 ([22])**.** *Let $X \subseteq \mathbb{P}^N$ be a projective variety, let $p_1, \ldots, p_s$ be generic points of X and let q be a generic point of $\langle p_1, \ldots, p_s \rangle$. Then*

$$\widehat{\mathbb{T}}_q\left(\sigma_s(X)\right) = \sum_{i=1}^{s} \widehat{\mathbb{T}}_{p_i}(X).$$

Remark 4.2. If X is a Segre-Veronese variety (or some other classically studied variety such as a Grassmann variety), one can represent its tangent space in a purely linear-algebraic manner. Let $u_i, v_i \in V_i \setminus \{0\}$ for each $i \in \{1, \ldots, k\}$. Consider the parametric curve given by $(u_1 + \varepsilon v_1)^{d_1} \otimes \cdots \otimes (u_k + \varepsilon v_k)^{d_k}$ for $\varepsilon \in K \setminus \{0\}$. Taking the derivative of this parametric curve at $\varepsilon = 0$ proves that the affine cone over the tangent space to $\operatorname{Seg}\left(\prod_{i=1}^k v_{d_i}\left(\mathbb{P}V_i\right)\right)$ at $p = \left[u_1^{d_1} \otimes \cdots \otimes u_k^{d_k}\right]$ in $\mathbb{P}\left(\bigotimes_{i=1}^k \operatorname{Sym}_{d_i} V_i\right)$ is given by

$$\widehat{\mathbb{T}}_p\left(\operatorname{Seg}\left(\prod_{i=1}^k v_{d_i}\left(\mathbb{P}V_i\right)\right)\right) = \sum_{i=1}^{k} u_1^{d_1} \otimes \cdots \otimes u_i^{d_i-1} V_i \otimes \cdots \otimes u_k^{d_k}.$$

Thus $\widehat{\mathbb{T}}_p\left(\operatorname{Seg}\left(\prod_{i=1}^k v_{d_i}\left(\mathbb{P}V_i\right)\right)\right)$ can be represented by an $s \cdot \left(\sum_{i=1}^k n_i + 1\right) \times \prod_{i=1}^k \binom{n_i+d_i}{d_i}$ matrix. To compute the dimension of $\mathbb{T}_q\left(\operatorname{Seg}\left(\prod_{i=1}^k v_{d_i}\left(\mathbb{P}V_i\right)\right)\right)$, we select s points $p_1, \ldots, p_s \in \operatorname{Seg}\left(\prod_{i=1}^k v_{d_i}\left(\mathbb{P}V_i\right)\right)$, construct the matrix representing $\widehat{\mathbb{T}}_{p_i}(X_{\mathbf{n},\mathbf{d}})$, stack these matrices and then compute the rank of the resulting matrix. If the rank of this matrix is equal to

$$\min\left\{s \cdot \left(\sum_{i=1}^k n_i + 1\right), \prod_{i=1}^k \binom{n_i + d_i}{d_i}\right\},$$

then we can conclude that $\sigma_s\left(\operatorname{Seg}\left(\prod_{i=1}^k v_{d_i}\left(\mathbb{P}V_i\right)\right)\right)$ has the expected dimension by semi-continuity.

4.2. **Inductive approach.** Let $X \subset \mathbb{P}^N$ be a non-singular variety of dimension n and let $X' \subset X$ be an n'-dimensional subvariety of X. Suppose that $\mathbb{P}^k$ is the linear subspace spanned by X'. Consider the projection π from $\mathbb{P}^k$ to the complementary linear subspace $\mathbb{P}^{N-k-1}$. We denote by X'' the projection of X from $\mathbb{P}^k$ to $\mathbb{P}^{N-k-1}$. If p lies in X', then $\pi(\mathbb{T}_p(X))$ is an $(n - n' - 1)$-dimensional linear subspace of $\mathbb{P}^{N-k-1}$. Suppose now that p lies in $X \setminus \mathbb{P}^k$ and let L_p be the intersection of $\mathbb{T}_p(X)$ with $\mathbb{P}^k$. Then $\pi(\mathbb{T}_p(X))$ is a linear subspace of $\mathbb{P}^{N-k-1}$ that is isomorphic to $\mathbb{T}_{\pi(p)}(X'')$. For example, if $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$ is a Segre variety and if $X' = \mathbb{P}^{n_1'} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$, then $X'' = \mathbb{P}^{n_1 - n_1' - 1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. Let $p = [v_1 \otimes v_2 \otimes v_3] \in X$. Then the linear spaces L_p and $\pi(\mathbb{T}_p(X))$ are isomorphic to $\mathbb{P}^{n_1'} \times \{[v_2]\} \times \{[v_3]\}$ and $\mathbb{P}^{n_1 - n_1' - 1} \times \{[v_2]\} \times \{[v_3]\}$ respectively.

Let $p_1, \ldots, p_s$ be generic points of X. Suppose that $p_1, \ldots, p_{s'} \in X'$ and that $p_{s'+1}, \ldots, p_s \in X \setminus X'$. Let

$$P' = \sum_{i=1}^{s'} \mathbb{T}_{p_i}(X') + \sum_{i=s'+1}^{s} L_i$$

and let

$$P'' = \sum_{i=1}^{s} \pi(T_{p_i}(X)) + \sum_{i=s'+1}^{s} \mathbb{T}_{\pi(p_i)}(X'').$$

If P' and P'' have the expected dimensions and if P' and P'' have the same "abundancy", i.e., either $P' = \mathbb{P}^k$ and $P'' = \mathbb{P}^{N-k-1}$ or P' and P'' are both proper linear subspaces of $\mathbb{P}^k$ and $\mathbb{P}^{N-k-1}$ respectively, then a standard application of Terracini's lemma shows that $\sigma_s(X)$ has the expected dimension. Thus $\dim \sigma_s(X)$ can be computed by induction on dimension and degree. We summarize how X' and X'' can be chosen in the same class of X in the list below.

	X	X'	X''
Segre	$\prod_{i=1}^{k} \mathbb{P}^{n_i}$	$\mathbb{P}^{n'_1} \times \prod_{i=2}^{k} \mathbb{P}^{n_i}$	$\mathbb{P}^{n_1-n'_1-1} \times \prod_{i=2}^{k} \mathbb{P}^{n_i}$
Grassmann	$\mathbb{G}(k,n)$	$\mathbb{G}(k,n-1)$	$\mathbb{G}(k-1,n)$
	$\mathbb{G}(2,n)$	$\mathbb{G}(2,n-6)$	Not applicable
Veronese	$v_d(\mathbb{P}^n)$	$v_d(\mathbb{P}^{n-1})$	$v_{d-1}(\mathbb{P}^n)$
Segre -Veronese	$\prod_{i=1}^{k} v_{d_i}(\mathbb{P}^{n_i})$	$v_{d_1}(\mathbb{P}^{n_1-1}) \times \prod_{i=2}^{k} v_{d_i}(\mathbb{P}^{n_i})$	$v_{d_1-1}(\mathbb{P}^{n_1}) \times \prod_{i=2}^{k} v_{d_i}(\mathbb{P}^{n_i})$
	$\mathbb{P}^{n_1-1} \times \prod_{i=2}^{k} v_{d_i}(\mathbb{P}^{n_i})$	$\mathbb{P}^{n'_1} \times \prod_{i=2}^{k} v_{d_i}(\mathbb{P}^{n_i})$	$\mathbb{P}^{n_1-n'_1-1} \times \prod_{i=2}^{k} v_{d_i}(\mathbb{P}^{n_i})$
	$\mathbb{P}^m \times v_2(\mathbb{P}^n)$	$\mathbb{P}^m \times v_2(\mathbb{P}^{n-2})$	Not applicable

5. Results

In the previous section, we considered an inductive approach for secant varieties of classically studied varieties. This inductive approach combined with Terracini's lemma has already allowed us to establish the existence of large classes of non-defective secant varieties from a small set of initial cases. In this section, we will review several results of the inductive approach on secant varieties for the Segre case and for the Grassmann case as current evidence for Conjectures 3.2 and 3.3. The results on secant varieties for the Segre-Veronese case will be summarized in the sequel paper.

5.1. **Segre varieties.** For $s \leq 6$, the classification of defective s^{th} secant varieties of Segre varieties is complete [3]. Using a Montecarlo technique combined with Terracini's lemma (see Theorem 4.1), we have verified that there are no balanced defective s^{th} secant varieties ($s \leq 8$) other than the known cases. In addition, Bryan Wilson and the author have implemented the inductive approach of [3] within the framework of Research Experience for Undergraduates Undecided about Pursuing Science (REU^2) [1] and checked that there are no defective s^{th} secant varieties of Segre varieties $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$ ($n_1 \leq n_2 \leq n_3 \leq 20$) other than the exceptions listed in Conjecture 3.2.

Three-factor Segre varieties. As was mentioned in Subsection 4.2, our approach allows one to reduce the computation of the dimension of $\sigma_s(X)$ to the computation of the dimensions of the secant varieties of a collection of three-factor Segre varieties. Thus completing the classification of defective Segre varieties will be necessary as the initial step towards the resolution of Conjecture 3.2.

Question 5.1. *Do all defective varieties of the form $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$ fall into one of the following cases {unbalanced, $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n$ with n even}?*

Perfect Segre varieties. The Segre variety $X = \prod_{i=1}^{k} \mathbb{P}^{n_i}$ is said to be *numerically perfect* if $s = (\prod_{i=1}^{k}(n_i+1))/(1+\sum_{i=1}^{k} n_i)$ is an integer and that X is *perfect* if $\sigma_s(X)$ has the expected dimension.

Question 5.2. *Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. If X is numerically perfect and balanced, then is X perfect?*

With the use of the approach in Section 4.2 combined with computer calculations, the balanced, numerically perfect cases with $n_1 \leq n_2 \leq n_3 \leq 100$ can be shown to be perfect.

Many copies of $\mathbb{P}^n$. Let $\underline{R}(n_1, \ldots, n_k)$ be the least integer such that a generic tensor in $\bigotimes_{i=1}^{k} V_i$ has tensor rank $\leq \underline{R}(n_1, \ldots, n_k)$. The integer $\underline{R}(n_1, \ldots, n_k)$ is called the *typical rank* of $\bigotimes_{i=1}^{k} V_i$ and can be used to estimate the maximal value of the rank of tensors in $\bigotimes_{i=1}^{k} V_i$. The problem of finding the typical rank of $\bigotimes_{i=1}^{k} V_i$ is equivalent to the problem of finding a positive integer s such that $\sigma_s(X)$ fills $\mathbb{P}^N$. So we have the following inequality:

$$\underline{R}(n_1, \ldots, n_k) \geq \left\lceil \frac{\prod_{i=1}^{k}(n_i+1)}{\left(1+\sum_{i=1}^{k} n_i\right)} \right\rceil. \tag{5.1}$$

[1] REU^2 is a program for freshman and sophomore students considering pursuing a science and/or engineering field of study. This program is supported by Experimental Program to Stimulate Competitive Research (EPSCoR) in Idaho.

An interesting subproblem is to prove the non-defectivity of many copies of $\mathbb{P}^n$. This is analogous to the following theorem that was proved by Lickteig in 1985:

Theorem 5.3 ([18])**.** *For all $n \neq 3$, $\dim \sigma_s(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n) = \min\{s(3n+1)-1,\ (n+1)^3-1\}$.*

In particular, the problem of determining the typical rank for the spaces that include the multiplication of square matrices has been completely solved. It is very natural to ask the question: "What is the typical rank $\underline{R}(n^k)$?". From Inequality (5.1) it follows that

$$\left\lceil \frac{(n+1)^k}{nk+1} \right\rceil \leq \underline{R}(n^k) = \underline{R}(\overbrace{n,\dots,n}^{k}).$$

Ottaviani, Peterson and the author proved the following theorem in [3]:

Theorem 5.4. *$\underline{R}(n^k)$ is asymptotically equivalent to $\frac{(n+1)^k}{nk+1}$ as $n \to \infty$ and $k \to \infty$.*

When $k \geq 3$, it is strongly believed that there are only a finite number of defective Segre varieties of the form

$$(\mathbb{P}^n)^k = \overbrace{\mathbb{P}^n \times \cdots \times \mathbb{P}^n}^{k}.$$

We believe that $(\mathbb{P}^2)^3$ and $(\mathbb{P}^1)^4$ are the only defective cases.

Question 5.5. *Let $k \geq 3$. Other than $(\mathbb{P}^2)^3$ and $(\mathbb{P}^1)^4$, is every Segre variety of the form $(\mathbb{P}^n)^k$ non-defective?*

5.2. **Grassmann varieties.** In [19], McGillivray used a probabilistic approach to compute the dimension of the s^{th} secant variety of $\mathbb{G}(k,n)$ for each (s,k,n) with $2 \leq k \leq \lceil \frac{n-1}{2} \rceil$ and verified that Conjecture 3.3 holds for $n \leq 14$. This computation has been recently extended to $n \leq 16$ by Baur, Draisma and de Graaf [6].

Monomial approach. In order to provide further evidence, Ottaviani, Peterson and the author used a "monomial" technique to generalize Theorem 2.1 of [12].

Theorem 5.6 ([2])**.** *If $3(s-1) \leq n-k$ and if $k \geq 2$, then*

$$\dim \sigma_s(\mathbb{G}(k,n)) = s((k+1)(n-k)+1)-1.$$

Ideas from error correcting codes were then used to strengthen this monomial approach to classify the defective s^{th} secant varieties of Grassmann varieties for small s.

Theorem 5.7 ([2])**.** *Suppose $k \geq 2$. Then:*

(i) *$\sigma_3(\mathbb{G}(k,n))$ has the expected dimension except for $(k,n) = (2,6)$, $(3,7)$.*
(ii) *$\sigma_4(\mathbb{G}(k,n))$ has the expected dimension except for $(k,n) = (2,8)$, $(3,7)$.*
(iii) *$\sigma_s(\mathbb{G}(k,n))$ has the expected dimension for $s = 2, 5, 6$.*

Typical alternating tensor rank. Let $\underline{R}(2,n)$ be the rank of a generic alternating tensor in $\wedge^2\mathbb{C}^n$. In [14], Ehrenborg proved that $\underline{R}(2,n)$ is bounded asymptotically by $\frac{n^2}{12}$. Ottaviani, Peterson and the arthor improved this result by applying the approach in Subsection 4.2.

Theorem 5.8 ([2])**.** *Let $n \geq 9$ and let*

$$s_1(n) = \left\lfloor \frac{n^2}{18} - \frac{2n}{27} + \frac{170}{81} \right\rfloor \quad \text{and} \quad s_2(n) = \left\lceil \frac{n^2}{18} + \frac{7n}{27} - \frac{73}{81} \right\rceil.$$

Then $\sigma_s(\mathbb{G}(2,n))$ has the expected dimension for $s \leq s_1(n)$ and for $s_2(n) \leq s$. In particular, $\underline{R}(2,n)$ is asymptotically equivalent to $\frac{n^2}{18}$ as $n \to \infty$.

References

[1] H. Abo and M. C. Brambilla, *On the dimensions of secant varieties of Segre-Veronese varieties*, arXiv:0912.4342.

[2] H. Abo, G. Ottaviani and C. Peterson, *Non-defectivityof Grassmannians of planes*, accepted for publication in J. Algebraic Geom.

[3] H. Abo, G. Ottaviani and C. Peterson, *Induction for secant varieties of Segre varieties*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 767–792.

[4] S. Abrescia, *About defectivity of certain Segre-Veronese varieties*, Canad. J. Math. **60** (2008), no. 5, 961–974.

[5] J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), no. 2, 201–222.

[6] K. Baur, J. Draisma and W. A. de Graaf, *Secant dimensions of minimal orbits: computations and conjectures*, Experiment. Math. **16** (2007), no. 2, 239–250.

[7] C. Bocci, *Special effect varieties in higher dimension*, Collect. Math. **56** (2005) no. 3, 299–326.

[8] E. Carlini and M. V. Catalisano, *Existence results for rational normal curves*, J. Lond. Math. Soc. (2) **76** (2007), no. 1, 73–86.

[9] E. Carlini and J. Chipalkatti, *On Waring's problem for several algebraic forms*, Comment. Math. Helv. **78** (2003), no. 3, 494–517.

[10] M. V. Catalisano, A. V. Geramita and A. Gimigliano, *On the ideals of secant varieties to certain rational varieties*, J. Algebra **319** (2008), no. 5, 1913–1931.

[11] M. V. Catalisano, A. V. Geramita and A. Gimigliano, *Higher secant varieties of Segre-Veronese varieties*, Projective varieties with unexpected properties, Walter de Gruyter GmbH & Co. KG, Berlin, 2005, 81–107.

[12] M. V. Catalisano, A. V. Geramita and A. Gimigliano, *Secant varieties of Grassmann varieties.* Proc. Amer. Math. Soc. **133** (2005), no. 3, 633–642

[13] C. Dionisi and C. Fontanari, *Grassman defectivity à la Terracini*, Matematiche **56** (2001), no. 2, 245-255.

[14] R Ehrenborg, *On apolarity and generic canonical forms*, J. Algebra **200** (1999), 167–194.

[15] R. L. Graham and N. J. A. Sloane, *Lower bounds for constant weight codes*, IEEE Trans. Inform. Theory **46** (1980), no. 1, 37–43.

[16] D. Grayson and M. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[17] H. Kaji, *Higher secant varieties of Grassmann varieties*, preprint, 1997.

[18] T. Lickteig, *Typical tensorial rank*, Linear Algebra Appl. **69** (1985), 95–120.

[19] B. McGillivray, *A probabilistic algorithm for the secant defect of Grassmann varieties*, Linear Algebra Appl. **418** (2006), no. 2–3, 708–718.

[20] G. Ottaviani, *Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited*, Vector bundles and low codimensional subvarieties: state of the art and recent developments, 315352, Quad. Mat., 21, Dept. Math., Seconda Univ. Napoli, Caserta, 2007.

[21] N. J. A. Sloane, `http://www.research.att.com/njas/codes/Andw.index.html`.

[22] A. Terracini, *Sulla rappresentazione delle forme quaternarie mediante somme di potenze di forme lineari*, Atti R. Accad. delle Scienze di Torino, vol. 51, 1915-16.

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