ON THE SECANT DEFECTIVITY OF SEGRE-VERONESE VARIETIES

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1. Introduction

Let $X \subseteq \mathbb{P}^N$ be a non-degenerate projective variety of dimension d. Then the s^{th} *secant variety* of X, denoted $\sigma_s(X)$, is the Zariski closure of the union of linear spans of s-tuples of points lying on X. The study of secant varieties has a long history. The interest in this subject goes back to the Italian school at the turn of the 20^{th} century. This topic has received renewed interest over the past several decades, mainly due to its increasing importance in an ever widening collection of disciplines including algebraic complexity theory [13, 26, 27], algebraic statistics [23, 22, 8], and combinatorics [29, 30].

The major questions surrounding secant varieties center around finding invariants of those objects such as dimension. A simple dimension count suggests that the expected dimension of $\sigma_s(X)$ is $\min\{s(d+1)-1, N\}$. We say that X has a *defective* s^{th} secant variety if $\sigma_s(X)$ does not have the expected dimension. In particular, X is said to be *defective* if X has a defective s^{th} secant variety for some s. The paper explores problems related to the classification of defective secant varieties of Sege-Veronese varieties. This is analogous to the celebrated theorem of Alexander and Hirschowitz [7], which asserts that higher secant varieties of Veronese varieties have the expected dimension (modulo a fully described list of exceptions). This work completed the *Waring problem* for polynomials which had remained unsolved for some time. There are corresponding conjecturally complete lists of defective secant varieties for Segre varieties [5] and for Grassmann varieties [25, 10]. Secant varieties of Segre-Veronese varieties are however less well-understood. In recent years, considerable efforts have been made to develop techniques to study secant varieties of Segre-Veronese varieties (see for example [18, 15, 14, 28, 18, 9, 16, 6]). But even the classification of defective two-factor Segre-Veronese varieties is still far from complete.

Very recently, the non-defectivity of two-factor Segre-Veronse varieties was systematically studied in [2], where the authors suggested that secant varieties of such Segre-Veronese varieties are not defective modulo the list of the well known exceptions. More precisely, they proposed the following conjecture:

Conjecture 1.1 ([3]). *Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, let $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$, and let $X_{\mathbf{n},\mathbf{d}}$ be the Segre-Veronese variety obtained from $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ embedded in multi-degree $\mathbf{d}$. Then $X_{\mathbf{n},\mathbf{d}}$ is defective if and only if:*

(i) $d_2 = 1$ *and* $n_2 \geq \binom{n_1+d_1}{d_1} - n_1 + 1$*;*
(ii) $\mathbf{n} = (1, n)$ *and* $\mathbf{d} = (2k, 2)$ *with* $k \geq 1$*;*
(iii) $\mathbf{n} = (4, 3)$*,* $(2, n)$ *with* n *odd and* $\mathbf{d} = (1, 2)$*;*
(iv) $\mathbf{n} = (1, 2)$*,* $(2, n)$ *and* $\mathbf{d} = (1, 3)$.

This paper discusses recent development towards the resolution of Conjecture 1.1. The main goal of this paper is to describe an inductive procedure that allows one to reduce computing the dimension of the secant variety of a Segre-Veronese variety to computing the dimension of a finite collection of "smaller" Segre-Veronese varieties.

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2. Castelnuouvo's Inequality

Let K denote an algebraically closed field of characteristic 0 and let V be an $(n+1)$-dimensional vector space over K. Throughout this paper, we denote by

- $V^* = \mathrm{Hom}_K(V, K)$ the dual of V,
- $\mathrm{Sym}_d V$ the d^{th} symmetric power of V,
- $\mathbb{P}V$ the projective space of V,
- $[v] \in \mathbb{P}V$ the equivalence class containing $v \in V \setminus \{0\}$,
- $\langle S \rangle$ the linear span of a subset S of $\mathbb{P}V$,
- $\widehat{X}$ the affine cone over a variety $X \subset \mathbb{P}V$ in V.
- $\mathbb{T}_p(X)$ the projective tangent space to a variety $X \subset \mathbb{P}V$ at a $p \in X$.

Let $v_d : \mathbb{P}V \to \mathbb{P}\mathrm{Sym}_d V$ be the d^{th} Veronese embedding, i.e., the map given by sending $[v]$ to $[v^d]$ and let $\mathrm{Seg} : \prod_{i=1}^k \mathbb{P}V_i \to \mathbb{P}\left(\bigotimes_{i=1}^k V_i\right)$ be the Segre embedding, i.e., the map given by sending $([v_1], \ldots, [v_k])$ to $[v_1 \otimes v_k]$, where $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and V_i is an $(n_i + 1)$-dimensional vector space over K for each $i \in \{1, \ldots, k\}$.

For a given $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we write $\mathbb{P}^{\mathbf{n}}$ for multi-projective space $\prod_{i=1}^k \mathbb{P}V_i$. Let $X_{\mathbf{n},\mathbf{d}}$ be the Segre-Veronese variety obtained from $\mathbb{P}^{\mathbf{n}}$ embedded in multi-degree $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$. In other words, $X_{\mathbf{n},\mathbf{d}} = \mathrm{Seg}\left(\prod_{i=1}^k v_{d_i} \mathbb{P}V_i\right) \subseteq \mathbb{P}\left(\bigotimes_{i=1}^k \mathrm{Sym}_{d_i} V_i\right)$.

We now explain how to translate the problem of computing the dimension of $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ into a question about the value of the Hilbert function of the ideal of s double points on $\mathbb{P}^{\mathbf{n}}$ at $\mathbf{d}$. First of all, we recall the well known theorem called *Terracini's lemma.*

Theorem 2.1 (Terracini's lemma)**.** *Let $p_1, \ldots, p_s$ be generic points of $X_{\mathbf{n},\mathbf{d}}$ and let q be a generic point of $\langle p_1, \ldots, p_s \rangle$. Then*

$$\widehat{\mathbb{T}}_q(\sigma_s(X_{\mathbf{n},\mathbf{d}})) = \sum_{i=1}^{s} \widehat{\mathbb{T}}_{p_i}(X_{\mathbf{n},\mathbf{d}}).$$

Note that $H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d}))$ can be identified with the set of hyperplanes in $\mathbb{P}\left(\bigotimes_{i=1}^k \mathrm{Sym}_{d_i} V_i\right)$. Since the condition that a hyperplane H of $\mathbb{P}\left(\bigotimes_{i=1}^k \mathrm{Sym}_{d_i} V_i\right)$ contains $\mathbb{T}_p(X_{\mathbf{n},\mathbf{d}})$ is equivalent to the condition that H intersects $X_{\mathbf{n},\mathbf{d}}$ in the first infinitesimal neighborhood of p, the elements of $H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_p^2(\mathbf{d}))$ can be viewed as hyperplanes containing $\mathbb{T}_p(X_{\mathbf{n},\mathbf{a}})$. Let Z be a collection of s double points on $\mathbb{P}^{\mathbf{n}}$ and let $\mathcal{I}_Z$ be its ideal sheaf. Terracini's lemma then implies that $\dim \sigma_s(X_{\mathbf{n},\mathbf{d}})$ equals to the value of the Hilbert function $h_{\mathbb{P}^{\mathbf{n}}}(Z, \cdot)$ of Z at $\mathbf{d}$, i.e.,

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{d}) = \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d})) - \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_Z(\mathbf{d})).$$

Therefore, proving that $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension is equivalent to proving that

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{d}) = \min\left\{ s\left(1 + \sum_{i=1}^{k} n_i\right),\ N(\mathbf{n}, \mathbf{d}) \right\},$$

where $N(\mathbf{n}, \mathbf{d}) = \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{d})) = \prod_{i=1}^k \binom{n_i + d_i}{d_i}$.

Definition 2.2. Let $\mathbf{n}, \mathbf{d} \in \mathbb{N}^k$, let s be a non-negative integer and let Z be a zero-dimensional subscheme of $\mathbb{P}^{\mathbf{n}}$. A triple $(\mathbf{n}; \mathbf{d}; Z)$ is said to be *subabundant* (resp. *superabundant*) if

$$\deg Z \le N(\mathbf{n}, \mathbf{d}) \text{ (resp. } \deg Z \ge N(\mathbf{n}, \mathbf{d})).$$

The triple $(\mathbf{n}; \mathbf{d}; Z)$ is said to be *equiabundant* if it is both subabundant and superabundant. We say that $T(\mathbf{n}; \mathbf{d}; Z)$ is *true* if $h_{\mathbb{P}^{\mathbf{n}}}(Z, .)$ has the expected value at $\mathbf{d}$. If Z is a collection of s double points, we write $T(\mathbf{n}; \mathbf{d}; s)$ instead of $T(\mathbf{n}; \mathbf{d}; Z)$ and $(\mathbf{n}; \mathbf{d}; s)$ instead of $(\mathbf{n}; \mathbf{d}; Z)$. We say that $T(\mathbf{n}; \mathbf{d})$ is true if $T(\mathbf{n}; \mathbf{d}; s)$ is true for every $s \ge 0$.

Suppose now that $d_1 \geq 2$. Denote by $\mathbf{n}'$ and $\mathbf{d}'$ the k-tuples $(n_1 - 1, n_2, \dots, n_k)$ and $(d_1 - 1, d_2, \dots, d_k)$ respectively. Given a $\mathbb{P}^{\mathbf{n}'} \subset \mathbb{P}^{\mathbf{n}}$, we have a short exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(\mathbf{d}') \to \mathcal{I}_Z(\mathbf{d}) \to \mathcal{I}_{Z\cap \mathbb{P}^{\mathbf{n}'}, \mathbb{P}^{\mathbf{n}'}}(\mathbf{d}) \to 0,$$

where $\widetilde{Z}$ is the residual scheme of Z with respect to $\mathbb{P}^{\mathbf{n}'}$. This short exact sequence gives rise to the so-called *Castelnuovo inequality*

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{d}) \geq h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{d}') + h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{d}).$$

Thus, we can conclude that

(a) $T(\mathbf{n}; \mathbf{d}'; \widetilde{Z})$ and $T(\mathbf{n}'; \mathbf{d}; Z \cap \mathbb{P}^{\mathbf{n}'})$ are both true and
(b) $(\mathbf{n}; \mathbf{d}'; \widetilde{Z})$ and $(\mathbf{n}'; \mathbf{d}; Z \cap \mathbb{P}^{\mathbf{n}'})$ have the same abundancy,

then $T(\mathbf{n}; \mathbf{d}; Z)$ is also the expected value and $(\mathbf{n}; \mathbf{d}; Z)$ has the same abundancy as $(\mathbf{n}; \mathbf{d}'; \widetilde{Z})$ and $(\mathbf{n}'; \mathbf{d}; Z \cap \mathbb{P}^{\mathbf{n}'})$. By semicontinuity, the Hilbert function of a general collection of s double points in $\mathbb{P}^{\mathbf{n}}$ has the expected value at $\mathbf{d}$. The problem that impedes this procedure is, however, that it may be impossible to arrange that Condition (b) is satisfied. We illustrate this in the following example:

Example 2.3. Let $\mathbf{n} = (2, 2)$, $\mathbf{d} = (4, 4)$, and $s = N(\mathbf{n}, \mathbf{d})/(\dim \mathbb{P}^{\mathbf{n}} + 1)$. Let $p_1, \dots, p_s$ be s points of $\mathbb{P}^{\mathbf{n}}$ and let $Z = \{p_1^2, \dots, p_s^2\}$. Then $(\mathbf{n}; \mathbf{d}; Z)$ is equiabundant, because $N(\mathbf{n}, \mathbf{d})/(\dim \mathbb{P}^{\mathbf{n}} + 1)$ is an integer. In order to show the truth of $T(\mathbf{n}; \mathbf{d}; Z)$, we would like to specialize a certain number of points to $\mathbb{P}^{\mathbf{n}'}$ in such a way that $(\mathbf{n}; \mathbf{d}'; \widetilde{Z})$ and $(\mathbf{n}'; \mathbf{d}; Z \cap \mathbb{P}^{\mathbf{n}'})$ have the same abundancy. This is impossible, because $N(\mathbf{n}', \mathbf{d})/(\dim \mathbb{P}^{\mathbf{n}'} + 1) = \binom{1+4}{4}\binom{2+4}{4}/(1 + 2 + 1) = 75/4$ is not an integer.

3. Méthode d'Horace différentielle

In this section, we will generalize the so-called *méthode d'Horace différentielle* introduced by Alexander and Hirschowitz to Segre-Veronese varieties, in order to give a way around the numerical obstacle as mentioned in Section 2.

Theorem 3.1 ([2]). *Let $d_1 \geq 3$. For a given non-negative integer s, let s' and ε be the quotient and remainder in the division of*

$$s\left(1 + \sum_{i=1}^{k} n_i\right) - \binom{n_1 + d_1 - 1}{d_1 - 1} \prod_{i=2}^{k} \binom{n_i + d_i}{d_i}$$

by $\sum_{i=1}^{k} n_i$. Suppose that $s' \geq \varepsilon$. If $T(\mathbf{n}'; \mathbf{d}; s')$, $T(\mathbf{n}; \mathbf{d}'; s - s')$ and $T(\mathbf{n}; \mathbf{d}''; s - s' - \varepsilon)$ are all true and if $(\mathbf{n}; \mathbf{d}''; s - s' - \varepsilon)$ is superabundant, i.e.,

$$(s - s' - \varepsilon)(\dim \mathbb{P}^{\mathbf{n}} + 1) \geq N(\mathbf{n}, \mathbf{d}''), \tag{3.1}$$

then $T(\mathbf{n}; \mathbf{a}; s)$ is also true.

Example 3.2. Let $\mathbf{n} = (2, 2)$, $\mathbf{d} = (4, 4)$, and $s = N(\mathbf{n}, \mathbf{d})/(\dim \mathbb{P}^{\mathbf{n}} + 1)$. Then $\mathbf{n}' = (1, 2)$, $\mathbf{d}' = (3, 4)$ and $\mathbf{d}'' = (2, 4)$, and thus $s' = 18$ and $\varepsilon = 3$. Hence we obtain

$$(s - s' - \varepsilon)(\dim \mathbb{P}^{\mathbf{n}} + 1) = (45 - 18 - 3) \cdot 5 = 120 > 90 = \binom{2 + 4 - 2}{2}\binom{2 + 4}{4} = N(\mathbf{n}, \mathbf{d}'').$$

Thus Theorem 3.1 implies that computing $\dim \sigma_{45}(X_{\mathbf{n},\mathbf{d}})$ can be reduced to computing the dimensions of $\sigma_{18}(X_{\mathbf{n}',\mathbf{d}})$, $\sigma_{27}(X_{\mathbf{n},\mathbf{d}'})$, and $\sigma_{24}(X_{\mathbf{n},\mathbf{d}''})$.

Remark 3.3. (i) Theorem 3.1 allows us to establish the non-defectivity of $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ by induction on $\mathbf{n}$ and $\mathbf{d}$.

(ii) Theorem 3.1 cannot be applied if $\mathbf{n} \leq (2, \dots, 2)$.

(iii) If $k = 2$ and if $d \geq 3$, then (3.1) holds for every $\mathbf{n} \in \mathbb{N}^k$. However, it is frequent that (3.1) does not hold if $\mathbf{d}$ is small, for example, one of the d_i's is 1. We thus need a different approach called the "splitting lemma" for such cases (see [3, 2] for the splitting lemma).

Below is an application of Theorem 3.1 and the splitting lemma.

Theorem 3.4 ([2])**.** *If $a, b \geq 3$, then the statement $T(n, 1; a, b)$ is true for every positive integer n.*

The following is an immediate consequence of Theorems 3.1 and 3.4:

Corollary 3.5 ([2])**.** *Suppose that $T(n, m; 3, 3)$, $T(n, m; 3, 4)$ and $T(n, m; 4, 4)$ are true for any n and m. Then $T(n, m; a, b)$ is true for any $a, b \geq 3$.*

This corollary leads naturally to the following conjecture:

Conjecture 3.6 ([2])**.** *If $\mathbf{d} \geq (3, 3)$, then $T(\mathbf{n}; \mathbf{d})$ is true for every $\mathbf{n} \in \mathbb{N}^2$.*

Many other examples of defective secant varieties of two factor Segre-Veronese varieties have also been discovered by several authors. In Table 1 below, we provide the list of such defective secant varieties.

	$\mathbf{n}$	$\mathbf{a}$	s	References
(1)	$(2, 2k+1)$	$(1, 2)$	$3k+2$	[28]
(2)	$(4, 3)$	$(1, 2)$	6	[15]
(3)	$(1, 2)$	$(1, 3)$	5	[21], [15]
(4)	$(1, n)$	$(2, 2)$	$n+2 \leq s \leq 2n+1$	[18], [16], [14]
(5)	$(2, 2)$	$(2, 2)$	$7, 8$	[18], [16]
(6)	$(2, n)$	$(2, 2)$	$\left\lfloor \frac{3n^2+9n+5}{n+3} \right\rfloor \leq s \leq 3n+2$	[18], [12]
(7)	$(3, 3)$	$(2, 2)$	$14, 15$	[18], [16]
(8)	$(3, 4)$	$(2, 2)$	19	[12]
(9)	$(n, 1)$	$(2, 2k)$	$kn+k+1 \leq s \leq kn+k+n$	[6]

TABLE 1.

Conjecture 3.7. *There exist no defective secant varieties of two-factor Segre-Veronese variety other than listed.*

A substantial amount of effort has been made to complete the list of defective secant varieties of two-factor Segre-Veronese varieties $X_{\mathbf{n},\mathbf{d}}$ for a given $(\mathbf{n}, \mathbf{d})$. Below we list the cases that have been fully understood. Please refer to Table 1 for the exceptions.

$\mathbf{n}$	$\mathbf{a}$	Exceptions	References
$(1, n)$	$(1, 2)$	None	[15]
$(2, n)$	$(1, 2)$	(1)	[3]
$(n, n-1)$	$(1, 2)$	(2)	[1]
(n, n)	$(1, 2)$	None	[1]
(k, n)	$(1, k+1)$	None	[17]
$(1, 2)$	$(1, b)$	(3)	[21]
$(n, 1)$	$(1, b)$	None	[20]
(m, n)	$(1, b)$ with $b \geq 3$ and $(m+n+1) \mid \binom{n+b}{b}$	None	[11]
$(n, 1)$	$(2, b)$	(9)	[6]
$(n, 1)$	$(3, b)$	(3)	[6]
$(1, 1)$	(a, b)	(9)	[17]
$(n, 1)$	(a, b) with $b \geq 3$	(9)	Theorem 3.4

TABLE 2.

4. Secant varieties of Segre-Veronese varieties embedded in multi-degree $(1,2)$

As was claimed in the previous two sections, the Castelnuovo's inequality and Theorem 3.1 enable one to establish the non-defectivity of $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ by induction on $\mathbf{n}$ and $\mathbf{d}$. Neither of them cannot be however applied to the case where $\mathbf{n}=(m,n)$ and $\mathbf{d}=(1,2)$, because it involves secant varieties of two-factor Segre varieties, most of which are known to be defective. To avoid this problem, one therefore needs an *ad hoc* approach. Recently, Chiara Brambilla and the author establish the existence of a large family of non-defective secant varieties of such Segre-Veronese varieties by double induction on m and n, where the authors found two functions $s_1(m,n) \leq s_2(m,n)$ such that $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension whenever $s \geq s_1(m,n)$ and whenever $s \leq s_2(m,n)$. More precisely, they proved the following theorem:

Theorem 4.1 ([3, 1])**.** *Let $\mathbf{n}=(m,n)$ and let $\mathbf{d}=(1,2)$. Suppose that $n \geq m-1$.*
(i) *Let $k=\lfloor n/2 \rfloor$ and let $s_1(m,n)$ be the function defined by*

$$s_1(m,n) = \begin{cases} (m+1)k-(m-2)(m+1)/2 & \textit{if } n \textit{ is even;} \\ (m+1)k-(m-3)(m+1)/2 & \textit{if } m \textit{ and } n \textit{ are odd;} \\ (m+1)k-[(m-3)(m+1)+1]/2 & \textit{otherwise.} \end{cases}$$

Then $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension for any $s \leq s_1(m,n)$.
(ii) *Let $s_2(m,n)$ be the function defined by*

$$s_2(m,n) = \begin{cases} \overline{s}(m,m)+(m+1)(n-m)/2 & \textit{if } 2|(n-m) \\ \overline{s}(m,m-1)+(m+1)(n-m+1)/2 & \textit{if } 2 \not| (n-m), \end{cases}$$

where $\overline{s}(m,m)=\lceil (m+1)\binom{m+2}{2}/(2m+1)\rceil$ and $\overline{s}(m,m-1)=\lceil (m+1)\binom{m+1}{2}/2m\rceil$. Then $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension for any $s \geq s_2(m,n)$.

Remark 4.2. Let m and n be integers with $n \geq m-1$ and let $s_1(m,n)$ be the function as defined in Theorem 4.1. Let $\underline{s}(m,n)=\left\lfloor (m+n+1)^{-1}(m+1)\binom{n+2}{2}\right\rfloor$. Then

$$\underline{s}(m,n) = \begin{cases} (m+1)\lfloor n/2\rfloor - \frac{(m-2)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor & \text{if } n \text{ is even;} \\ (m+1)\lfloor n/2\rfloor - \frac{(m-3)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)\lfloor n/2\rfloor - \frac{(m-3)(m+1)+1}{2} + \left\lfloor \frac{n+m^3+2}{2(m+n+1)} \right\rfloor & \text{otherwise.} \end{cases}$$

Define a function $r(m,n)$ as follows:

$$r(m,n) = \begin{cases} m^3-2m & \text{if } m \text{ is even and if } n \text{ is odd;} \\ (m-2)(m+1)^2/2 & \text{otherwise.} \end{cases}$$

Then $s_1(m,n)=\underline{s}(m,n)$ if $n \geq r(m,n)$.

Remark 4.3. Note that if $s(m+n+1) \leq (m+1)\binom{n+2}{2}$ and if $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension, then $\sigma_{s'}(X_{\mathbf{n},\mathbf{d}})$ also has the expected dimension for any $s' \leq s$. Likewise, if $s \geq (m+1)\binom{n+2}{2}$ and if $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension, then $\sigma_{s'}(X_{\mathbf{n},\mathbf{d}})$ also has the expected dimension for any $s' \geq s$. In particular, $X_{\mathbf{n},\mathbf{d}}$ is non-defective if and only if $\sigma_{\lfloor f(m,n)\rfloor}(X_{\mathbf{n},\mathbf{d}})$ and $\sigma_{\lceil f(m,n)\rceil}(X_{\mathbf{n},\mathbf{d}})$ have the expected dimension, where $f(m,n)=(m+n+1)^{-1}(m+1)\binom{n+2}{2}$. Thus, ideally, we would like $s_1(m,n)$ and $s_2(m,n)$ to satisfy $s_1(m,n) \leq s_2(m,n)-1$. While Theorem 4.1 does not reach this result, the difference $s_2(m,n)-s_1(m,n)$ is relatively small (it is asymptotically equivalent to m^2 as $m \to \infty$).

5. Computation with `Macaulay2`

Some of the results presented in this paper are partially based on computations in `Macaulay2`. (`Macaulay2` is a computer algebra system developed by Dan Grayson and Mike Stillman [24] for research in algebraic geometry and commutative algebra.) In this section, we demonstrate one of the key `Macaulay2` functions called `tangentToSecant`. In order to use this command, visit the author's website at http://www.webpages.uidaho.edu/~abo/programs.html and click "Secant varieties of Segre-Veronse varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$". Then the reader can download the file called SV.m2. Now save this file in the home directory.

The function tangentToSecant takes two k-tuples $\mathbf{n} = (n_1, \ldots, n_k)$ and $\mathbf{d} = (d_1, \ldots, d_k)$ of non-negative integers plus a positive integer s and it returns the tangent space to the s^{th} secant variety $\sigma_s(X_{\mathbf{n},\mathbf{d}})$ of $X_{\mathbf{n},\mathbf{d}}$ at a randomly chosen point of $\sigma_s(X_{\mathbf{n},\mathbf{d}})$. Let's start Macaulay2 and load the file as follows.

```
+ M2 --no-readline --print-width 97
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : load "SV.m2"
```

Let's try to compute $\dim \sigma_{45}(X_{\mathbf{n},\mathbf{d}})$ with $\mathbf{n} = (2,2)$ and $\mathbf{d} = (4,4)$. The input must be a list of two lists {2,2} and {4,4} and the positive integer 45.

```
i2 : T = tangentToSecant({{2,2},{4,4}},45);

                270        225
o2 : Matrix S     <--- S
```

The output T is the matrix that represents as the column space the affine cone $\widehat{\mathbb{T}}_q(\sigma_{45}(X_{\mathbf{n},\mathbf{d}}))$ over the projective tangent space to $\sigma_{45}(X_{\mathbf{n},\mathbf{d}})$ at a randomly selected point q of $\sigma_{45}(X_{\mathbf{n},\mathbf{d}})$. We can therefore find the dimension of $\widehat{\mathbb{T}}_q(\sigma_{45}(X_{\mathbf{n},\mathbf{d}}))$ computing the rank of T.

```
i3 : rank T

o3 = 225
```

This result implies that $\dim \widehat{\mathbb{T}}_q(\sigma_{45}(X_{\mathbf{n},\mathbf{d}})) = 225$, which means that $\sigma_{45}(X_{\mathbf{n},\mathbf{d}})$ has the expected dimension, because

$$45 \cdot (2+2+1) = \binom{2+4}{4}^2 = 225.$$

Note that we work in characteristic $p = 32003$. Our task is to check that a certain integer matrix has maximal rank. The Macaulay2 script determines that the matrix has maximal rank modulo p. Thus the result in characteristic zero follows from the openness of the maximal rank condition.

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