Causality Enforcement of High-Speed Interconnects via Periodic Continuations

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Abstract
Causality verification and enforcement is of great importance for performance evaluation of electrical interconnects. We present two techniques based on Kramers-Krönig dispersion relations, also called Hilbert transform relations, and construction of causal periodic continuations. The first method employes periodic polynomial continuations, while the second approach constructs Fourier continuations using a regularized singular value decomposition (SVD) method. Given a transfer function sampled on a bandlimited frequency interval, non-periodic in general, both approaches construct an accurate approximation on the given frequency interval by allowing the function to be periodic on an extended domain. This allows one to significantly reduce (for polynomial continuations) or even completely remove (for Fourier continuations) boundary artifacts that are due to the bandlimited nature of frequency responses. Using periodic continuations eliminates the necessity of approximating the transfer function behavior at infinity in order to compute Hilbert transform. The methods can be used to verify and enforce causality before the frequency responses are employed for macromodeling. The performance of the methods is analyzed and compared using moderately and highly non-smooth functions.

Key words
Causality, dispersion relations, high-speed interconnects, Hilbert transform, periodic continuation

I. Introduction

The design of high-speed interconnects requires systematic simulations at different levels in order to evaluate overall electrical system performance and avoid signal integrity problems [8]. To conduct such simulations, one needs suitable models of parts of the system that capture the relevant electromagnetic phenomena that may affect the signal and power quality. Such models are obtained either from direct measurements or full-wave electromagnetic simulations in the form of discrete port frequency responses that represent scattering, impedance, or admittance transfer functions or transfer matrices in scalar or multidimensional cases, respectively. Once frequency responses are available, a corresponding macromodel can be derived using, for example, the Vector Fitting technique [5]. However, if the data are contaminated by errors, it may not be possible to derive a good model. These errors may be due to a noise in case of direct measurements or roundoff and/or approximation errors occurring in full-wave numerical simulations. Besides, these data are typically available over a finite frequency range as discrete sets with a limited number of samples. All this may affect the performance of the macromodeling algorithm resulting in non-convergence or inaccurate models. Often the underlying cause of such behavior is the lack of causality in a given set of frequency responses [12].

A system is causal if the output cannot precede the input. For a linear time-translation invariant system with the impulse response function h, this condition implies that $h(t) = 0$ for $t < 0$. Causality can also be defined in the frequency domain. Denote by $H(w)$ the Fourier transform of $h(t)$, which is also called a transfer function of the system. Then the system is said to be causal if a frequency response given by the transfer function $H(w)$ satisfies the dispersion relations also known as Kramers-Krönig relations. The dispersion relations represent the fact that the real and imaginary parts of a causal function are related through Hilbert transform. The dispersion relations are very important in many areas of physics, science and engineering. Applications in electronics include reconstruction and correction [9] of measured data, delay extraction [6], estimation of optimal bandwidth and data density using causality checking [14] and causality enforcement techniques [10], [13], [1], [3], [2] that is the subject of the present study.

The Hilbert transform that relates the real and imaginary parts of a transfer function $H(w)$ is defined on the infinite domain which can be reduced to $[0, \infty)$ by symmetry properties of $H(w)$ in case if $h(t)$ is real. However, the frequency responses are usually available only over a finite length frequency interval, so either the domain has to be truncated or behavior of $H(w)$ for $w \to \infty$ needs to be approximated. Either approach brings an approximation error. Moreover, having bandlimited frequency responses causes significant boundary artifacts. One of the approaches used is to limit the class of transfer functions and assume that $H(w)$ is square integrable, which would require the function to decay at infinity. Then the domain can be truncated with a small error or asymptotic behavior of the transfer function at infinity can be obtained. In some cases, $H(w)$ is not bounded or even grows at infinity. Then the generalized dispersion relations can also be used [10] to decrease the dependence of $H(w)$ on high frequencies, and thus allow the domain truncation.

In this paper we take another approach and instead of approximating the behavior of $H(w)$ for large w or truncating the domain, we construct a periodic continuation of $H(w)$ by requiring the transfer function to be periodic and causal in an extended domain of finite length. In papers [1], [2], polynomial periodic continuations are introduced to make a transfer function periodic on an extended frequency interval and smooth at endpoints of the frequency interval. Once a periodic continuation is constructed, the spectrally accurate FFT/IFFT routines can be used to compute discrete Hilbert transform and enforce causality. The accuracy of the approach depends primarily on smoothness of the polynomial continuation or polynomial degree. The method allows one to significantly reduce the boundary artifacts compared to the computation of the Hilbert transform of the given data without any continuation, which is implemented in the function *hilbert* from the popular software Matlab. The advantage of the method is that it uses raw frequency responses on the original domain and, as a result, does not produce any spurious oscillations there, the method does not require a lot of data points and it is easy to implement. At the same time using low order polynomials allows one to detect only relatively large causality violations, for example, at the order of 10^{-3}, which may be enough for experimentally obtained data where high accuracy is not expected.

To improve the smoothness of periodic continuations and, thus, significantly reduce or completely eliminate boundary artifacts caused by using bandwidth limited frequency responses, an idea of approximating the transfer function by Fourier series in an extended domain was proposed in [3]. The approach allows one to obtain extremely accurate approximations of the given function on the original frequency interval. The causality conditions are imposed directly on Fourier coefficients that are then computed by solving an overdetermined linear system using regularized Singular Value Decomposition (SVD) method to address ill-conditioning of the problem. The method does not require using FFT/IFFT routines to compute Hilbert transform. The length of the extended domain can be varied to improve performance of the method. The advantage of the method is that it is extremely accurate and capable of detecting very small causality violations close to the machine precision, at the order of 10^{-13}, and the uniform reconstruction error of the transfer function on the entire original frequency interval can be achieved, so it does not have boundary artifacts reported in [11], [2].

In this paper we analyze and compare the performance of the causality verification methods based on periodic polynomial and Fourier continuations for moderately and highly non-smooth functions. We show that when frequency responses are represented by smooth or moderately non-smooth function, Fourier continuation based approach is much more accurate than the polynomial continuation based method and allows one to verify causality with the accuracy close to the machine precision as well as detect causality violations of amplitude close to the machine precision. When a transfer function is wildly oscillatory and has high slope regions, the polynomial continuation based method has relatively small error, while Fourier continuation based approach develops Gibbs like oscillations due to singularities in the transfer function.

The paper is organized as follows. Section II gives background on causality, dispersion relations and motivation for the proposed method. In Section III we briefly mention how causal polynomial continuations are constructed as well as present main steps in the derivation of causal spectrally accurate Fourier continuations using a truncated singular value decomposition (SVD). In Section IV, both methods are applied to moderately and highly non-smooth transfer functions to compare the performance of these methods. Finally, in Section V we present our conclusions.

II. Causality for Linear Time-Translation Invariant Systems

Consider a linear and time-invariant physical system with the impulse response $h(t, t')$ subject to a time-dependent input $f(t)$, to which it responds by an an output $x(t)$. Linearity of the system and time-translation invariance imply that the response $x(t)$ can be written as the convolution of the input $f(t)$ and the impulse response $h(t - t')$ [7]

$$x(t) = \int_{-\infty}^{\infty} h(t - t') f(t') dt' = h(t) * f(t). \tag{II.1}$$

Denote by

$$H(w) = \int_{-\infty}^{\infty} h(\tau)\, \mathrm{e}^{-iw\tau}\, d\tau \tag{II.2}$$

the Fourier transforms of $h(t)$, also called a transfer function. For multidimensional systems, transfer matrices are considered instead. The proposed methods can be generalized by applying them to each element of the transfer matrix.

The system is *causal* if the output cannot precede the input, i.e. if $f(t) = 0$ for $t < T$, the same must be true for $x(t)$.

This condition implies $h(\tau) = 0$, $\tau < 0$, and (II.2) becomes

$$H(w) = \int_0^\infty h(\tau)\, \mathrm{e}^{-iw\tau}\, d\tau. \quad \text{(II.3)}$$

Note that the integral in (II.3) is extended only over a half-axis, which implies that $H(w)$ has a regular analytic continuation in lower half of the w-plane.

If function $h(t)$ is square integrable, i.e. $\int_0^\infty |h(t)|^2 dt < C$, then $H(w)$ is also square integrable. Application of Cauchy's theorem allows one to express $H(w)$ for any point w on the real axis as [7]

$$H(w) = \frac{1}{\pi i} \fint_{-\infty}^{\infty} \frac{H(w')}{w - w'} dw', \quad \text{real } w, \quad \text{(II.4)}$$

where

$$\fint_{-\infty}^{\infty} = P \int_{-\infty}^{\infty} = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{w-\epsilon} + \int_{w+\epsilon}^{\infty} \right)$$

denotes Cauchy's principal value. Separating the real and imaginary parts of (II.4), we get

$$\operatorname{Re} H(w) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\operatorname{Im} H(w')}{w - w'} dw', \quad \text{(II.5)}$$

$$\operatorname{Im} H(w) = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\operatorname{Re} H(w')}{w - w'} dw'. \quad \text{(II.6)}$$

These expressions relate $\operatorname{Re} H$ and $\operatorname{Im} H$ of a causal function and they are called the dispersion relations or Kramers-Krönig relations. These formulas show that $\operatorname{Re} H$ and $\operatorname{Im} H$ of a causal function are not independent of each other and $\operatorname{Re} H$ at one frequency is related to $\operatorname{Im} H$ for all frequencies, and vice versa. Choosing either $\operatorname{Re} H$ or $\operatorname{Im} H$ completely determines another by causality. Recalling the definition of the Hilbert transform

$$\mathcal{H}[u(w)] = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{u(w')}{w - w'} dw',$$

we see that $\operatorname{Re} H$ and $\operatorname{Im} H$ form a Hilbert transform pair,

$$\operatorname{Re} H(w) = \mathcal{H}[\operatorname{Im} H(w)],\ \operatorname{Im} H(w) = -\mathcal{H}[\operatorname{Re} H(w)]. \quad \text{(II.7)}$$

In practice, the function $H(w)$ may not be square integrable or even bounded. It may also grow as $O(w^n)$, when $|w| \to \infty$, $n = 0, 1, 2, \ldots$. Then dispersion relations with subtractions can be used to modify the integrand and make it less dependent on high frequencies. This approach is used in [7], [10] to verify causality though it does not allow one to remove completely boundary artifacts that are due to the absence of out of band frequency responses.

We take an alternative approach motivated by an example of the periodic function $H(w) = \mathrm{e}^{-iaw}$, $a > 0$, that is not square integrable but still satisfies dispersion relations (II.5), (II.6). In practice the transfer function $H(w)$ given by frequency responses at the discrete set of values over a bandlimited interval is not periodic in general. The absence of out-of-band frequency responses causes significant boundary artifacts since dispersion relations require integration over the infinite domain. To overcome this problem, we construct causal periodic continuations of $H(w)$ in an extended domain of finite length either using polynomial continuations or Fourier continuations. These methods are explained in the next section.

III. **Causal periodic continuations**

The values of a transfer function $H(w)$ obtained either from numerical computations or direct measurements, are available in a discrete form on a finite frequency interval $[w_{min}, w_{max}]$, where $w_{min} \geq 0$. When $w_{min} = 0$ or $w_{min} > 0$, we have baseband or bandpass case, respectively. Since equations (II.5), (II.6) are homogeneous in the frequency variable, we can rescale $[w_{min}, w_{max}]$ to $[a, 0.5]$ using the transformation $x = \frac{0.5}{w_{max}} w$ for convenience, where $a = 0.5 \frac{w_{min}}{w_{max}} \geq 0$. The time domain impulse function $h(t)$ is often real-valued. Then the real and imaginary parts of $H(w)$, as the Fourier transform of $h(t)$, are even and odd functions, respectively. This implies that the values of the transfer function $H(x)$ are available on the unit interval $x \in [-0.5, 0.5]$ by spectrum symmetry if $a = 0$ or on $[-0.5, -a] \cup [a, 0.5]$ for $a > 0$. Hilbert transform relations (II.7) allow one to choose either $\operatorname{Re} H$ or $\operatorname{Im} H$ and then determine the other one by causality. We fix $\operatorname{Re} H$, an even function, and we want to reconstruct $\operatorname{Im} H$, an odd function. This choice avoids a possibility of having an unknown constant on the right hand side of (II.5) that is the limit of $H(w)$ at infinity [9].

A. *Polynomial periodic continuation*

Starting from the rescaled transfer function $H(x) = \operatorname{Re} H(x) + i \operatorname{Im} H(x)$ whose values are available at $x_j \in [-0.5, 0.5]$, $j = 1, \ldots, N$, (baseband case), the idea is to construct a polynomial continuation of $\operatorname{Re} H$, denoted by $\mathcal{C}_m(\operatorname{Re} H)$, that is periodic in an extended domain of length $b > 1$. This new function is defined in $[-\frac{b}{2}, \frac{b}{2}]$, equals $\operatorname{Re} H$ in $[-0.5, 0.5]$ and given by an mth degree polynomial $P_m(x)$, m is even, outside $[-0.5, 0.5]$. The polynomial P_m is computed by requiring continuity of $\mathcal{C}_m(\operatorname{Re} H)$ and its corresponding derivatives at $x = \pm 0.5$. Using higher m produces smoother continuation. The period b of the continuation $\mathcal{C}_m(\operatorname{Re} H)$ is usually chosen as twice as big as the length of the original domain, i.e. $b = 2$. Smaller values of b may be used when $H(x)$ develops high slopes in the boundary region.

In the bandpass case with $a > 0$, an even degree polynomial $\tilde{P}_m(x)$ can be constructed similarly to P_m to approximate the missing behavior of $H(x)$ in $[-a, a]$.

Once the polynomial periodic continuation $\mathcal{C}_m(\operatorname{Re} H)$ is available, equation (II.6) can be used to reconstruct $\operatorname{Im} H$. Since the right hand side of (II.6) is the convolution of $-\frac{1}{\pi w}$ and $\operatorname{Re} H(w)$, it can be computed using Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ via convolution theorem that gives

$$\operatorname{Im} H(x) = -\mathcal{F}^{-1} \left[\mathcal{F} \left[\frac{1}{\pi x} \right] \cdot \mathcal{F}[\mathcal{C}_m(\operatorname{Re} H)(x)] \right], \quad \text{(III.1)}$$

where $\mathcal{F}\left[\frac{1}{\pi x}\right] = i \operatorname{sgn}(k)$, k is the wave number, and we replaced $\operatorname{Re} H(w)$ with its continuation $\mathcal{C}_m(\operatorname{Re} H)$. Discrete Fourier transform and its inverse can be computed employing FFT/IFFT subroutines. We note that the direct application of these routines to the original non-periodic $H(w)$ (without periodic continuation) usually causes significant boundary artifacts since these routines are designed for periodic functions, and differences in values of $H(w)$ and its derivatives at endpoints are treated as discontinuities. The reconstruction of

$\operatorname{Im} H$ obtained from (III.1) is then compared to $\operatorname{Im} H$ on the original frequency interval and decision if the data are causal is made depending if the reconstruction error is smaller than some tolerance. For more details on the periodic polynomial continuation based method as well as how to choose the error threshold, please see [2]. The results of its application are shown in Section IV.

B. Fourier continuation

As shown in [1], [2], accuracy of a polynomial periodic continuation primarily depends on smoothness of polynomial extension including at the boundary of the given domain. With the fixed order of the continuation polynomial, the sensitivity of the method to causality violations is limited. Using higher degree polynomials requires approximation of higher order derivatives which increases complexity of the problem and may not be very convenient. To achieve much higher precision and simplicity, we construct an accurate Fourier series approximation of $H(x)$ by allowing the Fourier series to be periodic and causal in an extended domain. We start from the Fourier continuation of H that we denote by $\mathcal{C}_F(H)$ and define

$$\mathcal{C}_F(H)(x) = \sum_{k=-M/2+1}^{M/2} \alpha_k \, \mathrm{e}^{-\frac{2\pi i}{b} kx}, \qquad \text{(III.2)}$$

for even values of M, whereas when M is odd, the index k varies from $-\frac{M-1}{2}$ to $\frac{M-1}{2}$. We will take the number of Fourier coefficients M to be even for simplicity. As before, $b > 1$ is the period of approximation.

The functions $\phi_k(x) = \mathrm{e}^{-\frac{2\pi i k x}{b}}$, $k \in \mathbb{Z}$, form a complete orthogonal basis in $L_2[0, b]$ and satisfy $\overline{\phi_k}(x) = \phi_{-k}(x)$. Since $\mathcal{H}\{\mathrm{e}^{-iax}\} = i \operatorname{sgn}(a)\, \mathrm{e}^{-iax}$, one can separate real and imaginary parts in (III.2) and enforce causality by requiring $\operatorname{Im} \mathcal{C}_F(H)$ to be Hilbert transform of $-\operatorname{Re} \mathcal{C}_F(H)$. This procedure eliminates Fourier coefficients α_k with $k \le 0$. Evaluating $\mathcal{C}_F(H)(x)$ at $\{x_j\}$, $j = 1, \ldots, N$, $x_j \in [-0.5, 0.5]$, where the values of $H(x)$ are known, produces a system

$$\mathcal{C}_F(H)(x_j) = \sum_{k=1}^{M/2} \alpha_k \phi_k(x_j), \quad j = 1, \ldots, N, \qquad \text{(III.3)}$$

of N linear equations for unknowns α_k, $k = 1, \ldots, M/2$. Since Fourier coefficients α_k are real, system (III.3) can be solved either as a complex system, or, alternatively, real and imaginary parts can be separated to produce a real system and, thus, ensure that a numerical solution $\{\alpha_k\}$ is indeed real. We find that solving real formulation gives slightly more accurate results. The number of Fourier coefficients $M/2$ is recommended to be as twice as small as the number N of data, i.e. $M = N$ [4]. This creates an overdetermined linear system that needs to be solved in the least-squares sense. Because of high ill-conditioning of the problem that increases with M and does not depend of the function H itself, we use regularized Singular Value Decomposition (SVD) method to compute a minimum norm solution. The singular values below a tolerance ξ that is close to the machine precision are discarded. We set $\xi = 10^{-13}$. The choice of the length b of the extended domain follows the same guidelines as for the polynomial periodic continuation. We typically use $b = 2$. Smaller values of b can be chosen for functions having steep slopes in the boundary region. More details about this approach can be found in [3].

In the next section, we demonstrate and compare the performance of periodic polynomial and Fourier continuation based methods to verify causality of the given frequency responses, modeled by moderately and highly non-smooth functions. We also study sensitivity of these methods to causality violations.

IV. Numerical Examples

A. Two-pole Example

Consider a transfer function with two poles defined by

$$H(w) = \frac{r}{iw + p} + \frac{\bar{r}}{iw + \bar{p}}$$

with $r = 1 + 2i$, $p = 1 + 3i$. The poles of $H(w)$ are located in the upper half w-plane at $\pm 3 + i$, hence, this function is causal of a sum of two causal transforms. H is sampled on $[0, 10]$ GHz, values of $\operatorname{Re} H$ are reflected to negative frequencies using the spectrum symmetry. Then the frequency interval is rescaled to $[-0.5, 0.5]$. $\operatorname{Re} H$ is shown in the top panel of Fig. 1. Superimposed is its periodic 8th degree polynomial continuation $\mathcal{C}_8(\operatorname{Re} H)(x)$ with $b = 2$. In the bottom panel of Fig. 1 we plot $\operatorname{Im} H$ and its reconstruction $-\mathcal{H}[\mathcal{C}_8(\operatorname{Re} H)(x)]$ using the continuation. For comparison, we also show the result of applying Hilbert transform to $\operatorname{Re} H$ directly without any continuation, which is computed using Matlab built-in function *hilbert*. It is clear that agreement between $\operatorname{Im} H$ and $-\mathcal{H}[\mathcal{C}_8(\operatorname{Re} H)(x)]$ is much better than between $\operatorname{Im} H$ and $-\mathcal{H}[\operatorname{Re} H(x)]$, especially in the boundary region. To analyze the quality of reconstruction of $\operatorname{Im} H$ on

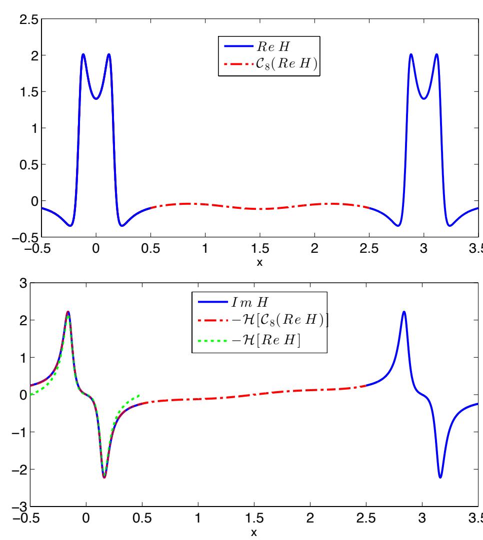

Fig. 1. Top panel: $\operatorname{Re} H(x)$ and its periodic 8th degree polynomial continuation $C_8(\operatorname{Re} H)(x)$ in Example IV-A with $N = 1000$, $b = 2$ shown on $[-0.5, 3.5]$. Bottom panel: $\operatorname{Im} H$ and its reconstructions $-\mathcal{H}[\mathcal{C}_8(\operatorname{Re} H)(x)]$ and $-\mathcal{H}[\operatorname{Re} H(x)]$ with and without periodic continuation, respectively.

the original frequency interval (rescaled), we study the error $E_{C,m}$ defined as

$$E_{C,m}(x) = \operatorname{Im} H + \mathcal{H}[\mathcal{C}_m(\operatorname{Re} H)(x)], \quad x \in [-0.5, 0.5].$$

For completeness we also introduce the error E of reconstruction of $\operatorname{Im} H$ from $\operatorname{Re} H$ without any continuation:

$$E(x) = \operatorname{Im} H + \mathcal{H}[\operatorname{Re} H(x)], \quad x \in [-0.5, 0.5].$$

The graph of the error $E_{C,8}$ using 8th degree polynomial continuation is shown in Fig. 2. For comparison, we plot in

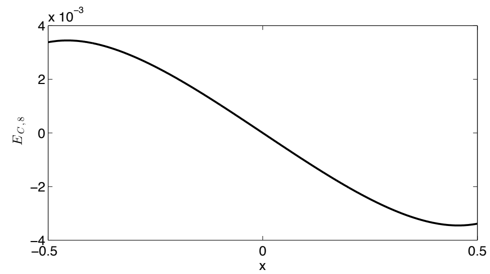

Fig. 2. Reconstruction error $E_{C,8}(x)$ in Example IV-A.

Fig. 3 the reconstruction errors $E_{C,m}$ with $m = 2$, 4, 6, 8 together with the error E when no continuation is used. These results show that using periodic continuation significantly reduces the reconstruction error. Moreover, the higher degree of the polynomial $P_m(x)$ is, i.e., the smoother continuation is, the smaller reconstruction error is. With $m = 8$, this error is decreased by about 15 times compared to the error when no continuation is used. Using higher degree polynomials will decrease the error even more as error analysis in [2] indicates.

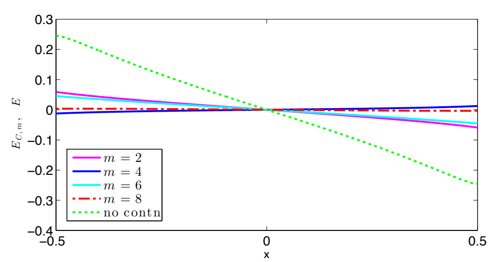

Fig. 3. Errors $E_{C,m}$ in reconstruction of $\operatorname{Im} H$ in Example IV-A using continuation $\mathcal{C}_m(\operatorname{Re} H(x))$ with polynomials of degree $m = 2$, 4, 6, 8 and $b = 2$, and E from the direct computation of Hilbert transform (without continuation) using Matlab function *hilbert*.

Now we apply the Fourier continuation based method to the same example using the same number of sample points $N = 1000$ and $b = 2$. We use $M = N$. The errors

$$E_R(x) = \operatorname{Re} H(x) - \operatorname{Re} \mathcal{C}_F(H)(x),$$

$$E_I(x) = \operatorname{Im} H(x) - \operatorname{Im} \mathcal{C}_F(H)(x)$$

in reconstruction of $\operatorname{Re} H$ and $\operatorname{Im} H$, respectively, using either complex or real formulation (see [3] for more details) of system (III.3) are shown in Fig. 4. Results indicate that using Fourier continuation approach allows one to get extremely small reconstruction error for a causal function. With $M = 1000$, the errors are on the order of 10^{-10}, while with $M = 1500$ the error reduces to 10^{-14}, which is close to the machine precision. Moreover, it is clearly seen that the reconstruction

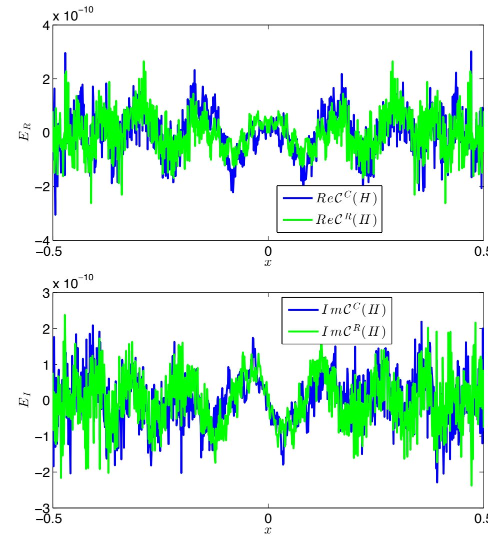

Fig. 4. Logarithmic plot of reconstruction errors $E_R(x)$ and $E_I(x)$ in example IV-A on $[-0.5, 0.5]$ with $M = N = 1000$, $b = 2$ using Fourier continuation based method. Superscripts ^{C} and ^{R} correspond to solving system (III.3) in complex and real formulation.

error is uniform on the entire interval $[-0.5, 0.5]$ as opposed to the error obtained using polynomial continuation based approach [2] or generalized dispersion relations [11], that was greater in the boundary region.

We also tested both periodic continuation based methods to detect causality violations. This is done by imposing a localised artificial causality violation modeled by Gaussian $a \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$ of small amplitude a, standard deviation σ, centered at x_0 and added to $\operatorname{Re} H$, while keeping $\operatorname{Im} H$ unchanged. We find that polynomial periodic continuation method allows one to detect causality violations of amplitude up to 10^{-3}, while Fourier spectral continuation approach is much more sensitive to such violations and perturbations of size up to 10^{-13} can be detected successfully by observing very pronounced spikes in the reconstruction errors $E_R(x)$ and $E_I(x)$ [3].

B. Highly Non-smooth Example

Next we analyze the performance of the periodic continuation based method applied to a non-smooth frequency response function $H(w)$ that was artificially created in [5] to test the performance of the Vector Fitting algorithm. The transfer function H is given in the Laplace domain as a rational function of order 18 in the pole-residue form

$$H(s) = \sum_{n=1}^{18} \frac{c_n}{s - a_n},$$

where c_n are residues, a_n poles with 2 poles being real and the rest being in complex conjugate pairs located in the left half plane. The values of poles and residues can be found in [5]. Since all poles are in the left half plane, the system that function $H(s)$ represents is passive, hence it is automatically causal [12]. To convert H to frequency domain, we use substitution $s = \mathrm{e}^{-i\frac{\pi}{2}} w = -iw$, which is rotation of w-plane by $\pi/2$ in the clockwise direction, so that poles of $H(s)$ in the left half s-plane correspond to poles of $H(w)$ in the lower w-plane.

In Fig. 5 we show $\operatorname{Re} H(x)$, its periodic 8th degree polynomial continuation $C_8(\operatorname{Re} H)(x)$ as well as reconstructions of $\operatorname{Im} H$ with and without continuations. The error $E_{C,8}(x)$ does not exceed the value of 2 and it is about twice smaller than the error E without a continuation. Since the values of $H(x)$ are quite large in this example, the relative error is only 0.7%. The Fourier continuation approach does not work well in this example. The Fourier approximations for both $\operatorname{Re} H$ and $\operatorname{Im} H$ have very pronounced Gibbs like oscillations due to the presence of discontinuities in H. By increasing the number of Fourier coefficients $M/2$ with the fixed resolution N, it is possible to construct an approximation that will be very close to H at the given frequency points but this approximation becomes insensitive to causality violations. This is a reason for which the number of Fourier coefficients cannot be arbitrary and needs to be related to the given resolution, i.e. $M = N$.

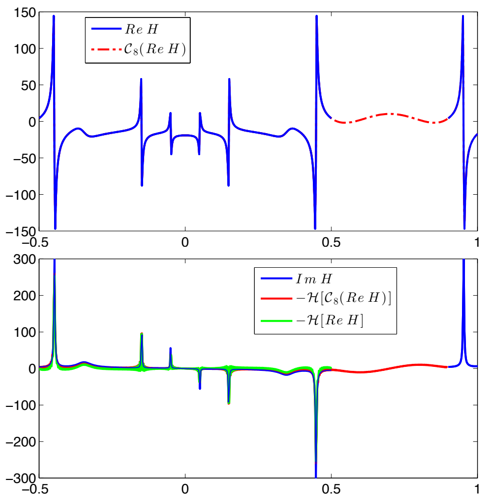

Fig. 5. Top panel: $\operatorname{Re} H(x)$ and its periodic 8th degree polynomial continuation $C_8(\operatorname{Re} H)(x)$ in Example IV-B with $N = 1000$, $b = 2$ shown on $[-0.5, 3.5]$. Bottom panel: $\operatorname{Im} H$ and its reconstructions $-\mathcal{H}[\mathcal{C}_8(\operatorname{Re} H)(x)]$ and $-\mathcal{H}[\operatorname{Re} H(x)]$ with and without periodic continuation, respectively.

For more examples of test transfer functions including simulated frequency responses as well as non-causal transfer functions, please see [2], [3].

V. Conclusions

We present two numerical methods based on construction of periodic continuations that allow one to verify and enforce if necessary the causality of tabulated frequency responses. The first method uses polynomial periodic continuation, while the second calculates accurate Fourier series approximations of transfer functions, not periodic in general. Both approaches require continuations to be periodic in an extended domain of finite length. For periodic continuation based method, starting from the real part of the transfer function and computing its continuation, the corresponding imaginary part is reconstructed by causality using the Kramers-Krönig dispersion relations that require real and imaginary parts of the transfer function to be a Hilbert transform pair. This is done by using FFT/IFFT routines. For Fourier continuation based method, the causality is imposed directly on Fourier coefficients that are computed in the least squares sense using truncated SVD method and without FFT/IFFT. Both approaches eliminate the necessity of approximating the behavior of the transfer function at infinity. The polynomial continuation based method allows one to significantly reduce boundary artifacts that are due to the lack of out-of band frequency responses, while Fourier continuation approach is able to completely remove them and produces uniform small errors close to the machine precision. The latter method is also more sensitive to causality violations. For highly non-smooth transfer functions, polynomial continuation based method handles better discontinuities, while Fourier continuation based approach suffers from Gibbs like oscillations. The methods are applicable to both baseband and bandpass regimes.

VI. Acknowledgments

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