Causality Enforcement Via Periodic Continuations for Package and On-Chip Interconnects

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***Abstract*—Causality verification and enforcement is of great importance for performance evaluation of electrical interconnects. We introduce a new technique based on Kramers-Krönig dispersion relations, also called Hilbert transform relations, and a construction of causal Fourier continuations using a regularized singular value decomposition (SVD) method. Given a transfer function sampled on a bandlimited frequency interval, non-periodic in general, this approach constructs highly accurate Fourier series approximations on the given frequency interval by allowing the function to be periodic on an extended domain. The causality is enforced spectrally, which eliminates the necessity to approximate the transfer function behavior as frequency goes to infinity in order to compute Hilbert transform. The performance of the proposed method is tested using a non-smooth frequency response function. The obtained results demonstrate an excellent accuracy and reliability of the proposed technique. We also compare the accuracy of the new method with the polynomial periodic continuation approach developed by authors in [1], [2].**

***Index Terms*—Causality, dispersion relations, Kramers-Krönig relations, Fourier continuation, periodic continuation, polynomial continuation, Hilbert Transform, least-squares solution, regularized SVD, high-speed interconnects.**

I. Introduction

The design of high-speed interconnects requires systematic simulations at different levels in order to evaluate overall electrical system performance and avoid signal integrity problems. To conduct such simulations, one needs suitable models of parts of the system that capture the relevant electromagnetic phenomena that may affect the signal and power quality. Such models are obtained either from direct measurements or full-wave electromagnetic simulations in the form of discrete port frequency responses that represent scattering, impedance, or admittance transfer functions or transfer matrices in multidimensional cases, respectively. Once frequency responses are available, a corresponding macromodel can be derived using, for example, the Vector Fitting (VF) technique [3]. However, if the data are contaminated by errors, it may not be possible to derive a good model. These errors may be due to a noise in case of direct measurements or roundoff and/or approximation errors occurring in full-wave numerical simulations. Besides, these data are typically available over a finite frequency range as discrete sets with a limited number of samples. All this may affect the performance of the macromodeling algorithm resulting in non-convergence or inaccurate models. Often the underlying cause of such behavior is the lack of causality in a given set of frequency responses [4].

A system is causal if a frequency response given by the transfer function $H(w)$ satisfies the dispersion relations also known as Kramers-Krönig relations [5], [6]. The dispersion relations characterize the causality in the frequency domain. They represent the fact that the real and imaginary parts of a causal function are related through Hilbert transform. The dispersion relations are very important in many areas of physics, science and engineering. Applications in electronics include reconstruction and correction [7] of measured data, delay extraction [8], estimation of optimal bandwidth and data density using causality checking [9] and causality enforcement techniques [10], [11] that is the subject of the present study.

The Hilbert transform that relates the real and imaginary parts of a transfer function $H(w)$ is defined on the infinite domain which can be reduced to $[0,\infty)$ by symmetry properties of $H(w)$. However, the frequency responses are usually available over a finite length frequency interval, so either the domain has to be truncated or behavior of $H(w)$ for $w\to\infty$ needs to be approximated. Either approach brings an approximation error. To handle this problem, one may assume that $H(w)$ is square integrable, which would require the function to decay at infinity. Then the domain can be truncated with some error. The generalized dispersion relations can also be used [10] to decrease the dependence of $H(w)$ on high frequencies, and thus allow the domain truncation.

In this paper we take another approach and instead of approximating the behavior of $H(w)$ for large w or truncating the domain, we construct a periodic continuation of $H(w)$ by requiring the transfer function to be periodic and causal in an extended domain of finite length. In papers [1], [2], polynomial periodic continuations are used to make a transfer function periodic on an extended frequency interval. Once a periodic continuation is constructed, the spectrally accurate FFT/IFFT routines can be used to compute discrete Hilbert transform and enforce causality. The accuracy of the method depends primarily on the order of the polynomial, which implies the smoothness of the polynomial continuation at the end points of the given frequency domain. This allows one to reduce the boundary artifacts compared to the performance of the function *hilbert* from the popular software Matlab that also implements discrete Hilbert transform. At the same time, having a higher polynomial order in the extended domain results in oscillations that affect the computation of the Hilbert transform. The advantage of the method is that it uses raw

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frequency responses on the original domain and, as a result, does not produce any spurious oscillations there, the method does not require a lot of data points and it is easy to implement. At the same time it allows one to detect only relatively large causality violations.

In the current work we implement the idea of periodic continuations by approximating a function by Fourier series in an extended domain. The approach allows one to obtain extremely accurate approximations of the given function on the original interval. The causality conditions are imposed directly on Fourier coefficients, which eliminates the necessity of approximating the behavior of the transfer function as the frequency tends to infinity similar to polynomial continuations used in [2], and does not require computation of Fourier Transform via FFT/IFFT. Even though the continuations exhibit high frequency oscillations in the extended domain, they do not significantly affect the quality of approximation in the boundary regions after the causality conditions are imposed. The length of the extended domain can be varied to minimize their effect. The advantage of the method is that it is capable of detecting very small causality violations close to the machine precision, at the order of 10^{-13}, and the uniform reconstruction error of the transfer function on the entire original frequency interval can be achieved, so it does not have boundary artifacts compared to *hilbert* Matlab function or polynomial continuation.

The paper is organized as follows. Section II gives background on causality, dispersion relations and motivation for the proposed method. In Section III we briefly mention how polynomial continuations are constructed as well as present the main steps in the derivation of causal spectrally accurate Fourier continuations using a truncated singular value decomposition (SVD). In Section IV, both methods are applied to a non-smooth transfer function in the residue-pole form to test the performance of the method. Finally, in Section V we present our conclusions.

II. Causality for Linear Time-Translation Invariant Systems

Consider a linear and time-invariant physical system with the impulse response $h(t,t')$ subject to a time-dependent input $f(t)$, to which it responds by an an output $x(t)$. Linearity of the system and time-translation invariance imply that the response $x(t)$ can be written as the convolution of the input $f(t)$ and the impulse response $h(t-t')$ [12]

$$x(t) = \int_{-\infty}^{\infty} h(t-t')f(t')dt' = h(t) * f(t). \tag{II.1}$$

Denote by

$$H(w) = \int_{-\infty}^{\infty} h(\tau)\, \mathrm{e}^{-iw\tau}\, d\tau \tag{II.2}$$

the Fourier transforms of $h(t)$, also called a transfer function.

The system is *causal* if the output cannot precede the input, i.e. if $f(t) = 0$ for $t < T$, the same must be true for $x(t)$. This condition implies $h(\tau) = 0$, $\tau < 0$, and (II.2) becomes

$$H(w) = \int_{0}^{\infty} h(\tau)\, \mathrm{e}^{-iw\tau}\, d\tau. \tag{II.3}$$

Note that the integral in (II.3) is extended only over a half-axis, which implies that $H(w)$ has a regular analytic continuation in lower half of the w-plane.

If function $h(t)$ is square integrable, i.e. $\int_0^\infty |h(t)|^2 dt < C$, then $H(w)$ is also square integrable [13], [14]. Application of Cauchy's theorem allows one to express $H(w)$ for any point w on the real axis as [12]

$$H(w) = \frac{1}{\pi i} \fint_{-\infty}^{\infty} \frac{H(w')}{w'-w} dw', \quad \text{real } w, \tag{II.4}$$

where

$$\fint_{-\infty}^{\infty} = P\int_{-\infty}^{\infty} = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{w-\epsilon} + \int_{w+\epsilon}^{\infty}\right)$$

denotes Cauchy's principal value. Separating the real and imaginary parts of (II.4), we get

$$\operatorname{Re} H(w) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\operatorname{Im} H(w')}{w'-w} dw', \tag{II.5}$$

$$\operatorname{Im} H(w) = -\frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{\operatorname{Re} H(w')}{w'-w} dw'. \tag{II.6}$$

These expressions relating $\operatorname{Re} H$ and $\operatorname{Im} H$ are called the dispersion relations or Kramers-Krönig dispersion relations after Krönig [6] and Kramers [5] who derived the first known dispersion relation for a causal system of a dispersive medium. These formulas show that $\operatorname{Re} H$ at one frequency is related to $\operatorname{Im} H$ for all frequencies, and vice versa. Recalling that the Hilbert transform is defined

$$\mathcal{H}[u(w)] = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{u(w')}{w-w'} dw',$$

we see that $\operatorname{Re} H$ and $\operatorname{Im} H$ are Hilbert transforms of each other, i.e.

$$\operatorname{Re} H(w) = -\mathcal{H}[\operatorname{Im} H(w)],\ \operatorname{Im} H(w) = \mathcal{H}[\operatorname{Re} H(w)]. \tag{II.7}$$

The function $H(w)$ may not satisfy the assumption of square integrability and it may be only bounded or even behave like $O(w^n)$, when $|w| \to \infty$, $n = 0, 1, 2, \ldots$. Then dispersion relations with subtractions can be used to verify causality [12], [10]. We take an alternative approach motivated by an example of the periodic function $H(w) = \mathrm{e}^{iwt}$, $t > 0$, that is not square integrable but still satisfies dispersion relations (II.5), (II.6). The transfer function $H(w)$ is not periodic in general and typically available only over a bandlimited interval. Direct application of dispersion relations (II.5), (II.6) does not work especially in the boundary regions mainly because the high-frequency behavior of $H(w)$ is missing, unless data decay to zero at the boundary like for low-pass filters. To overcome this problem, we construct causal periodic continuations of $H(w)$ in an extended domain either using polynomial continuations or Fourier continuations. These methods are explained in the next section.

III. Causal periodic continuations

The values of a transfer function $H(w)$ obtained either from numerical computations or direct measurements, are available in a discrete form on a finite frequency interval $[w_{min}, w_{max}]$, where $w_{min} \geq 0$. For simplicity we consider the baseband case $w_{min} = 0$. The approach can be generalized to bandpass case $w_{min} > 0$ as well [2], [15]. Since equations (II.5), (II.6) are homogeneous in the frequency variable, we can rescale $[0, w_{max}]$ to $[0, 0.5]$ using the transformation $x = 0.5/w_{max}$ for convenience. The time domain impulse function $h(t)$ is often real-valued. Then the real and imaginary parts of $H(w)$, as the Fourier transform of $h(t)$, are even and odd functions, respectively. This implies that the values of the transfer function $H(x)$ are available on the unit interval $x \in [-0.5, 0.5]$ by spectrum symmetry. Hilbert transform relations (II.7) allow one to choose either $\operatorname{Re} H$ or $\operatorname{Im} H$ and then determine the other one by causality. We fix $\operatorname{Re} H$, an even function, and we want to reconstruct $\operatorname{Im} H$, an odd function. If the error between $\operatorname{Im} H$ and its reconstructed version is small, the system is considered causal. Otherwise, it is non-causal.

A. Polynomial periodic continuation

The idea of a polynomial continuation of $\operatorname{Re} H$ is to construct an even function, denoted by $\mathcal{C}_p(\operatorname{Re} H)$, that is periodic in an extended domain with period $b > 1$, where b can be varied to achieve the best results. This function is a polynomial $P(x)$ of even degree outside the given domain $[-0.5, 0.5]$ and equals $\operatorname{Re} H$ in $[-0.5, 0.5]$. The symmetry of $P(x)$ implies that coefficients of odd powers of x are zero. The coefficients of even powers of x are computed by requiring continuity of $\mathcal{C}_p(\operatorname{Re} H)$ and its corresponding derivatives at $x = \pm 0.5$ [2]. Once $\mathcal{C}_p(\operatorname{Re} H)$ is constructed, FFT/IFFT routines can be used to compute the Hilbert transform of $\mathcal{C}_p(\operatorname{Re} H)$ and obtain a causal continuation of $\operatorname{Im} H$, which is then compared to the given $\operatorname{Im} H$ on the original frequency domain. The performance of the method is shown in Section IV.

B. Fourier continuation

As shown in [1], [2], accuracy of a polynomial periodic continuation primarily depends on smoothness of polynomial extension including at the boundary of the given domain, but using higher degree polynomials causes oscillations in the extended domain, which affect the computation of the Hilbert transform. This limits sensitivity of the method to causality violations. To achieve much higher precision, we construct an accurate Fourier series approximation of $H(x)$ by allowing the Fourier series to be periodic and causal in an extended domain. The result is the Fourier continuation of H that we denote by $\mathcal{C}_F(H)$, and it is defined by

$$\mathcal{C}_F(H)(x) = \sum_{k=-M/2+1}^{M/2} \alpha_k \, \mathrm{e}^{\frac{2\pi i}{b} kx}, \qquad \text{(III.1)}$$

for even values of M, whereas when M is odd, the index k varies from $-\frac{M-1}{2}$ to $\frac{M-1}{2}$. We will take the number of Fourier coefficients M to be even for simplicity. As before, $b > 1$ is the period of approximation.

The functions $\phi_k(x) = \mathrm{e}^{\frac{2\pi i k x}{b}}$, $k \in \mathbb{Z}$, form a complete orthogonal basis in $L_2[0, b]$ and satisfy $\overline{\phi_k}(x) = \phi_{-k}(x)$. Since $\mathcal{H}\{\mathrm{e}^{iwx}\} = -\operatorname{sgn}(x)\,\mathrm{e}^{iwx}$, one can separate real and imaginary parts in (III.1) and enforce causality by requiring $\operatorname{Im} \mathcal{C}_F(H)$ be Hilbert transform of $\operatorname{Re} \mathcal{C}_F(H)$. This procedure eliminates Fourier coefficients α_k with $k < 0$ and we can write

$$\mathcal{C}_F(H)(x) = \sum_{k=0}^{M/2} \alpha_k \phi_k(x).$$

Evaluating $\mathcal{C}_F(H)(x)$ at $\{x_j\}$, $j = 1, \ldots, N$, $x_j \in [-0.5, 0.5]$, where the values of $H(x)$ are known, produces a system of linear equations for α_k. The system can be either overdetermined or underdetermined depending on the values of M and N and it is highly ill-conditioned. We can solve this system in the least-squares sense and compute a minimum norm solution using the truncated SVD method. For more details, please see [15]. As we demonstrate in the next section, this method allows one to achieve extremely high accuracy of approximation of the transfer function on the original frequency interval even for non-smooth functions by retaining enough Fourier coefficients.

IV. Example: verification of causality

We validate both methods using a non-smooth frequency response function $H(w)$ that was artificially created in [3] to test the performance of the Vector Fitting algorithm. The transfer function H is given in the Laplace domain as a rational function of order 18 in the pole-residue form

$$H(s) = \sum_{n=1}^{18} \frac{c_n}{s - a_n}, \qquad \text{(IV.1)}$$

where c_n are residues, a_n poles with 2 poles being real and the rest being in complex conjugate pairs located in the left half plane. The values of poles and residues can be found in [3]. Since all poles are in the left half plane, the system that function $H(s)$ represents is passive, hence it is automatically causal [4]. We use this example to test accuracy of causality verification. To convert H to frequency domain, we use substitution $s = \mathrm{e}^{-i\frac{\pi}{2}}\, w = -iw$, which is rotation of w-plane by $\pi/2$ in the clockwise direction, so that poles of $H(s)$ in the left half s-plane correspond to poles of $H(w)$ in the lower w-plane.

In Fig. 1 we show $\operatorname{Re} H(x)$ together with its 4th order polynomial extension using $b = 1.4$ and $\operatorname{Im} H(w)$ with superimposed Hilbert transform of periodically continued $\mathcal{C}_p(\operatorname{Re} H(x))$. $H(w)$ was sampled on $[0, 8 \cdot 10^3]$ Hz and then rescaled to $[0, 0.5]$. For comparison, we also plot the Hilbert transform of the original $\operatorname{Re} H$ (without periodic extension) obtained using Matlab function *hilbert*. The absolute error with both approaches, plotted in the left panel of Fig. 2, indicates that the polynomial continuation method has about 10 times smaller overall reconstruction error especially compared in the boundary regions.

Next we apply Fourier continuation method to the same function (IV.1). With this approach, the complex valued function $\mathcal{C}_F(H)(x)$ is causal by construction. The graphs of $\operatorname{Re} H(x)$ and $\operatorname{Im} H(x)$ together with their causal Fourier continuations are shown in Fig. 3. We use $N = 250$ points that fixes the resolution of $H(x)$. Note that this function is non-smooth because of the poles, so fixing the number of points regularizes the function by discretization. As can be seen from Fig. 3, the given $\operatorname{Re} H(x)$ and $\operatorname{Im} H(x)$ and

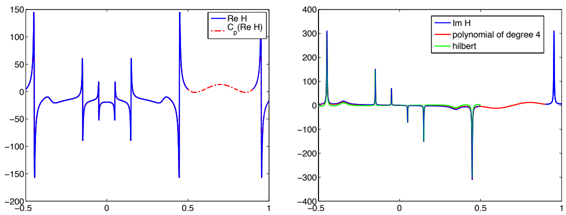

Fig. 1. Left panel: $\mathrm{Re}\, H$ with its $4th$ order polynomial continuation $\mathcal{C}_p(\mathrm{Re}\, H)$, $b = 1.4$, $N = 1500$. Right panel: $\mathrm{Im}\, H$ with Hilbert transform $\mathcal{H}[\mathcal{C}_p(\mathrm{Re}\, H)]$ of the continuation (red curve) and $\mathcal{H}[\mathrm{Re}\, H]$ computed using Matlab *hilbert* function (green curve).

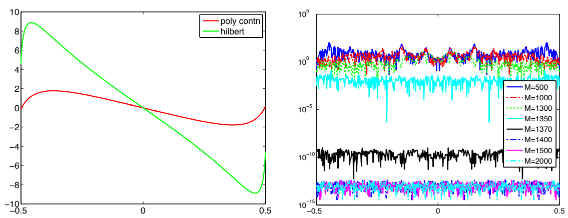

Fig. 2. Left panel: absolute error in approximating $\mathrm{Im}\, H$ using 4th order polynomial continuation $\mathcal{C}_p(\mathrm{Re}\, H)$ compared to Matlab function *hilbert* applied to $\mathrm{Re}\, H$ directly. Right panel: absolute error (semilogy plot) in approximation of $\mathrm{Re}\, H$ using causal Fourier continuations with $N = 250$ and M varied from 500 to 2000.

their causal Fourier continuations are almost indistinguishable. The absolute error in reconstruction of both $\mathrm{Re}\, H(x)$ and $\mathrm{Im}\, H(x)$ with $M \geq 1400$ is at the order of 10^{-13}, and it is uniform on the entire interval unlike polynomial continuation method. We also show in the right panel of Fig. 2 that the number M of Fourier coefficients can be used to achieve the maximum accuracy, and as M increases with fixed resolution, the accuracy increases until it reaches some limit close to the machine precision.

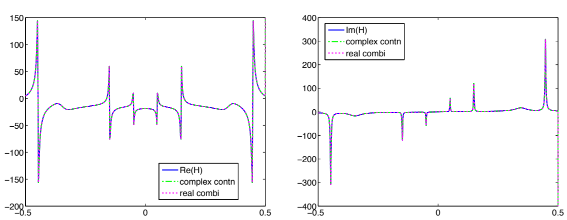

Fig. 3. Real and imaginary parts of the transfer function $H(w)$ and their causal Fourier continuations with $M = 2000$, $N = 250$, $b = 1.4$ shown on $[-0.5, 0.5]$. Top panel: real part. Bottom panel: imaginary part.

We also added a small Gaussian perturbation (not shown here) that models causality violation and find that the approach allows one to detect tiny causality violations at the order of $10^{-13} - 10^{-12}$. For more details about causal Fourier continuation method and its performance, please see [15].

V. Conclusions

We present a numerical method based on construction of SVD-based Fourier continuations that allows one to verify and enforce if necessary the causality of tabulated frequency responses. This is done by calculating accurate Fourier series approximations of transfer functions, not periodic in general, and allowing the Fourier series to be periodic in an extended domain. The causality is imposed directly on Fourier coefficients using the Kramers-Krönig dispersion relations that require real and imaginary parts of the transfer function to be a Hilbert transform pair. This eliminates the necessity of approximating the behavior of the transfer function as the frequency tends to infinity, which is known to be a source of significant errors in computation of the Hilbert transform defined on an infinite domain (or semi-infinite due to spectrum symmetry) with data available only on a finite bandwidth. The Fourier coefficients are computed by solving an oversampled or undersampled regularized least squares problem via a truncated SVD method to have the ill-conditioning of the system under control. The method is applicable to both baseband and bandpass regimes and does not require data points to be equally spaced. The proposed technique is applied to a non-smooth artificially created transfer function. The results demonstrate an excellent performance of the method to detect even small causality violations unlike polynomial continuation approach.

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