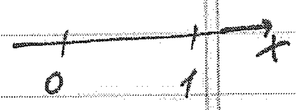
GREEN'S FUNCTION FOR A LINEAR 2nd ORDER DIFFERENTIAL EQUATION. SUPERPOSITION.

Let $L(u)$ be a linear 2^{nd} order differential operator on functions $u(x)$, $0 \le x \le 1$.

not a step function

For given homogeneous boundary conditions, for example, $u(0)=0$, $u(1)=0$, Green's function $u = G(x, \xi)$ satisfies

$$Lu = \delta(x-\xi) \qquad \text{for a fixed } 0 < \xi < 1$$



$$u(0)=0; \quad u(1)=0$$

Ex $u'' + 5u' + 2u = 0$ $\qquad L = \frac{d^2}{dx^2} + 5\frac{d}{dx} + 2$

$$\underbrace{x^2 u'' + 3x u' + e^x u}_{Lu} = 0 \qquad L = x^2 \frac{d^2}{dx^2} + 3x\frac{d}{dx} + e^x$$

Green's function has an important property: the solution of inhomogeneous problem

$$Lu = f(x)$$

w/ the same BCs: $u(0)=0; \; u(1)=0$

is given by <u>superposition integral</u>

$$u(x) = \int_0^1 f(\xi)\, G(x,\xi)\, d\xi$$

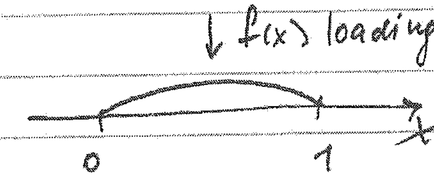
<u>Ex</u> Steady state temperature distribution.

$$u_t = k u_{xx}$$

Steady-state $\Rightarrow \frac{\partial u}{\partial t} = 0 \Rightarrow u_{xx} = 0$

$$\left\{ \begin{aligned} &Lu = -\frac{d^2u}{dx^2} = f(x) \\ &u(0)=0, \; u(1)=0 \end{aligned} \right.$$

represent static deflection of a string that is clamped at both end points



If BCs had different constants

$$u(0)=a, \quad u(1)=b$$

we can use a linear transformation to reduce problem w/ BCs

$$u \rightarrow w$$

$$u(x) = w(x) + \alpha + \beta x$$

α, β: determined to satisfy $\overset{\text{homog. BCs}}{w(0)=0,\ w(1)=0}$

at $x=0$: $\underbrace{u(0)}_{=a} = \underbrace{w(0)}_{=0} + \alpha + \beta \cdot 0 \Rightarrow \boxed{\alpha = a}$

at $x=1$: $\underbrace{u(1)}_{=b} = \underbrace{w(1)}_{=0} + \underset{=a}{\alpha} + \beta \cdot 1 \Rightarrow \boxed{\beta = b-a}$

$\Rightarrow$ the homogenizing transformation is

$$u(x) = w(x) + \underbrace{a + (b-a)x}_{\text{linear in } x}$$

$$(\smile)'' = 0 \Rightarrow u'' = w''$$

$\Rightarrow w(x)$ satisfies the same DE but homog. BCs.

$\Rightarrow$ wlg we can consider problem w/ homog. BCs.

$$-\frac{d^2u}{dx^2} = f(x)$$

$u(0)=0;\ u(1)=0$

$$\underset{\text{general solution}}{u(x)} = \underset{\text{solution of homog. problem}}{u_h(x)} + \overset{\text{particular solution}}{u_p(x)}$$

$$-\frac{d^2y}{dx^2}=0$$

$y_h(x) = Ax+B$: complementary function

We can use the method of variation of parameters to find $y_p(x)$, particular solution.

Let

$$y_p(x) = u_1(x)\cdot x + u_2(x)$$

$u_1(x), u_2(x)$: functions to be determined

<u>Aside:</u>

(1) $$a_2(x)y'' + a_1(x)y' + a_0(x)y = R(x)$$

lin. indep.

$y_1(x)$, $y_2(x)$: solutions of associated homog. eq^2, i.e.

$$a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y_1 = 0$$

$$a_2(x)y_2'' + a_1(x)y_2' + a_0(x)y_2 = 0$$

$$y_h(x) = C_1y_1(x) + C_2y_2(x)$$

Assume

(2) $$y_p(x) = u_1(x)\cdot y_1(x) + u_2(x)\cdot y_2(x)$$

Substitute $y_p(x)$ into (1):

$$(a_0) \quad y_p = u_1 \cdot y_1 + u_2 \cdot y_2$$

$$(a_1) \quad y_p' = u_1 \cdot y_1' + u_2 \cdot y_2' + \{u_1' \cdot y_1 + u_2' \cdot y_2\}$$

$$(a_2) \quad y_p'' = \underset{\downarrow\, 0}{u_1 \cdot y_1''} + \underset{\downarrow\, 0}{u_2 \cdot y_2''} + u_1' \cdot y_1' + u_2' \cdot y_2' + \{u_1' \cdot y_1 + u_2' \cdot y_2\}'$$

$$a_1\{u_1' \cdot y_1 + u_2' \cdot y_2\} + a_2[u_1' \cdot y_1' + u_2' \cdot y_2'] + a_2\{u_1' \cdot y_1 + u_2' y_2\}' = R(x)$$

We have 1 eq^{n} for 2 unknowns $\{..\}$ & $[\ ]$

We have undetermined eq^{n} $\Rightarrow$ one of variables is a free parameter

Let $\{\ldots\} = 0$ $\Downarrow$ $[\ldots] = \dfrac{R(x)}{a_2}$

$$\text{Let} \quad \begin{cases} u_1' \cdot y_1 + u_2' \cdot y_2 = 0 \\ u_1' \cdot y_1' + u_2' \cdot y_2' = \dfrac{R(x)}{a_2(x)} \end{cases} \qquad (3)$$

To find $u_1(x)$, $u_2(x)$, we first solve (3) for $u_1'(x)$, $u_2'(x)$. Then we integrate to get $u_1(x)$, $u_2(x)$.

We can write (3) in matrix form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{R(x)}{a_2(x)} \end{pmatrix}$$

$$\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = W(y_1, y_2): \quad \text{Wronskian of } y_1, y_2$$

$\neq 0$ since y_1, y_2 are lin. indep.

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for our example,

$$u_p(x) = u_1(x)\cdot \underbrace{x}_{y_1(x)} + u_2(x)\cdot \underbrace{1}_{y_2(x)}$$

We solve for u_1', u_2' the following system

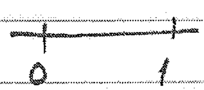
$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{f(x)}{-1} \end{pmatrix}$$

$$\underbrace{-}_{a_2(x) = -1} u''(x) = \underbrace{f(x)}_{R(x)}$$

$$x u_1' + u_2' = 0 \quad \Rightarrow \quad u_2' = -x u_1' = x f(x)$$

$$\boxed{u_1' = -f(x)} \qquad \boxed{u_2' = x f(x)}$$

Integrate to get u_1, u_2:



$$u_1(x) = \int_0^x u_1'(\xi)\,d\xi + \cancelto{0}{u_1(0)} = -\int_0^x f(\xi)\,d\xi$$

$$u_2(x) = \int_0^x u_2'(\xi)\,d\xi + \cancel{u_2(0)}^{\,0} = \int_0^x \xi f(\xi)\,d\xi$$

Hence,

$$u_p(x) = x \cdot u_1(x) + u_2(x) =$$

$$= -\int_0^x x f(\xi)\,d\xi + \int_0^x \xi f(\xi)\,d\xi = \int_0^x (\xi - x) f(\xi)\,d\xi$$

Then,

$$u(x) = \underbrace{Ax + B}_{u_h} + \underbrace{\int_0^x (\xi - x) f(\xi)\,d\xi}_{u_p}$$

Ex (Cont'd)

(1) $$-\frac{d^2u}{dx^2} = f(x)$$

(2) $u(0)=0$

(3) $u(1)=0$

Using the method of variation of parameters we showed that the general solution of eq^{n} (1) is

$$u(x) = Ax + \cancel{B}^{\,0} + \int_0^x (\xi - x) f(\xi)\, d\xi$$

BCs:

$$u(0)=0 \Rightarrow A\cdot 0 + B + \int_0^0 \ldots d\xi = 0 \Rightarrow \boxed{B=0}$$

$$u(1)=0 \Rightarrow A\cdot 1 + \int_0^1 (\xi - 1) f(\xi)\, d\xi = 0$$

$$\boxed{A = \int_0^1 (1-\xi) f(\xi)\, d\xi}$$

Substitute A and B into $u(x)$:

$$u(x) = \underbrace{\int_0^1}_{\int_0^x + \int_x^1} x(1-\xi) f(\xi)\, d\xi + \int_0^x (\xi - x) f(\xi)\, d\xi =$$

$$u(x) = \int_0^x \left[x(1-\xi) f(\xi) + (\xi - x) f(\xi) \right] d\xi +$$

$$+ \int_x^1 x(1-\xi) f(\xi)\, d\xi$$

$$[\dots] = \left(x(\cancel{1}-\xi) + (\xi - \cancel{x}) \right) f(\xi) = \left(-x\xi + \xi \right) f(\xi) =$$

$$= \xi (1-x) f(\xi)$$

$$u(x) = \int_0^x \xi(1-x) f(\xi)\, d\xi + \int_x^1 x(1-\xi) f(\xi)\, d\xi$$

Recall

$$u(x) = \int_0^1 f(\xi)\, G(x,\xi)\, d\xi$$

Hence, Green's function is

$$G(x,\xi) = \begin{cases} \xi(1-x), & 0 < \xi \leq x \\ x(1-\xi), & x \leq \xi < 1 \end{cases}$$

Now we will derive Green's function using its definition.

Green's function $G(x,\xi)$ is the solution of

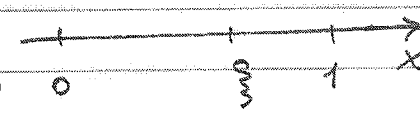
$Lu = \delta(x-\xi) \qquad 0 < \xi < 1$

$u(0)=0, \quad u(1)=0$

or in this example

(4) $$-\frac{d^2u}{dx^2} = \delta(x-\xi) \qquad \delta(x-\xi) = \begin{cases} 0 & x \neq \xi \\ \infty & x = \xi \end{cases}$$

$u(0)=0, \quad u(1)=0$



For a fixed ξ, consider eq.^{n} (4) on two intervals to the left and right from ξ: $0 \le x < \xi$ and $\xi < x \le 1$. Since $\delta(x-\xi)=0$ on both intervals $\Rightarrow \frac{d^2u}{dx^2}=0 \Rightarrow u(x)$ is a linear function.

$$u(x) = \begin{cases} Ax + B, & 0 \le x < \xi \\ Cx + D, & \xi < x \le 1 \end{cases}$$

Constants A, B, C, D may be different in general.

$u(0)=0 \Rightarrow A \cdot 0 + B = 0 \Rightarrow B = 0$

$u(1)=0 \Rightarrow C \cdot 1 + D = 0 \Rightarrow D = -C$

$$\therefore \quad u(x) = \begin{cases} Ax, & 0 \le x < \xi \\ C(x-1), & \xi < x \le 1 \end{cases}$$

$$-u'' = \delta(x-\xi)$$

infinite at $x = \xi$

$\Rightarrow$ u' has a finite jump at $x=\xi$

$\Rightarrow$ u is continuous at $x = \xi$

$$\Rightarrow \lim_{x\to\xi^-} u(x) = \lim_{x\to\xi^+} u(x)$$

$$\boxed{A\xi = C(\xi-1)}$$

We need one more condition for a jump across $\xi = x$. To derive it, we will integrate

$$-u'' = \delta(x-\xi)$$

over the small interval $\xi - \varepsilon \le x \le \xi + \varepsilon$

$$-\int_{\xi-\varepsilon}^{\xi+\varepsilon} u''(x)\,dx = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,dx = 1$$

$$-u'(x)\Big|_{x=\xi-\varepsilon}^{x=\xi+\varepsilon} = 1$$

$$-u'(\xi+\varepsilon) + u'(\xi-\varepsilon) = 1$$

$\lim_{\varepsilon \to 0}$:

$$\boxed{-u'(\xi^+) + u'(\xi^-) = 1}$$ jump condition

$$[f] = f(\xi^+) - f(\xi^-)$$

jump at ξ

$\therefore \quad -[u']\Big|_{\xi} = 1$: jump condition

Recall

$$u(x) = \begin{cases} Ax, & x < \xi \\ C(x-1), & x > \xi \end{cases}$$

$$u'(x) = \begin{cases} A, & x < \xi \\ C, & x > \xi \end{cases}$$

$$\Rightarrow u'(\xi^+) = C, \quad u'(\xi^-) = A$$

$$\Rightarrow -u'(\xi^+) + u'(\xi^-) = 1 \Rightarrow \boxed{-C + A = 1}$$

jump condition

Hence,

$$\begin{cases} A\xi = C(\xi-1) & \text{continuity} \\ -C+A=1 & \text{jump condition} \end{cases}$$

We solve this system to get

$$\boxed{A = 1-\xi, \quad C = -\xi}$$

Then

$$\underset{\overset{\shortparallel}{G(x,\xi)}}{u(x)} = \begin{cases} Ax = (1-\xi)x, & 0 \le x < \xi \\ C(x-1) = -\xi(x-1), & \xi < x \le 1 \end{cases}$$

or

$$G(x,\xi) = \begin{cases} (1-\xi)x, & 0 \le x < \xi \\ (1-x)\xi, & \xi < x \le 1 \end{cases} \quad \text{as before}$$

<u>Note</u>:

$$G(\xi, x) = G(x,\xi):$$ Green's function has symmetry property

<u>Reason</u>: $\delta(x-\xi) = \delta(\xi-x)$

In general, for 2^{nd} order linear operator

$$L(u) \equiv a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)u = \delta(x-\xi)$$

$a_2(x)$, $a_1(x)$, $a_0(x)$ are continuous on $[0,1]$

we have continuity condition

$$u(\xi^+) - u(\xi^-) = 0$$

and the jump condition

$$a_2(\xi)\left[u'(\xi^+) - u'(\xi^-)\right] = 1$$

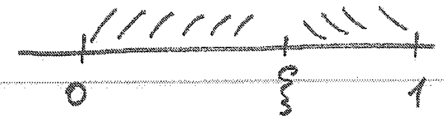
Example 2 Variable thermal diffusivity

Suppose the termal diffusivity K^2 (or we might have used K instead) varies linearly with x. In dimensionless variables, the Green's function for a steady-state temperature $u(x)$ in a conductor $0<x<1$ satisfies

$$(5) \quad -\frac{d}{dx}\left(x\frac{du}{dx}\right) = \delta(x-\xi), \qquad 0<\xi<1$$

$u(0) =$ finite (temperature is finite at $x=0$ when thermal diffusivity $K^2=0$

$u(1) = 0$



As before, we solve (5) on two intervals $0 \le x < \xi$ and $\xi < x \le 1$ where $\delta(x-\xi) = 0$; ξ is fixed.

$$u' = \frac{A}{x}$$

$$u = A \log x + B$$

$$u(x) = \begin{cases} A \log x + B, & 0 \le x < \xi \\ C \log x + D, & \xi < x \le 1 \end{cases}$$

$u(0)$ is finite $\Rightarrow A = 0$

$u(1) = 0 \Rightarrow C \cdot \underbrace{\log 1}_{\to 0} + D = 0 \Rightarrow D = 0$

$$\Rightarrow \quad u(x) = \begin{cases} B, & 0 \le x < \xi \\ C \log x, & \xi < x \le 1 \end{cases}$$

ex Variable thermal diffusivity

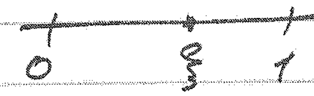
Green's function $G(x,\xi)$ satisfies

(1) $$-\frac{d}{dx}\left(x\frac{du}{dx}\right) = \delta(x-\xi)$$

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right) = -\frac{du}{dx} - x\frac{d^2u}{dx^2} = \overbrace{-x}^{a_2(x)}\frac{d^2u}{dx^2} - \frac{du}{dx}$$

(2) $u(0) = \text{finite}$

(3) $u(1) = 0$



$$u(x) = \begin{cases} A\ln x + B, & x<\xi \\ C\ln x + D, & x>\xi \end{cases}$$

$u(0) < \infty \Rightarrow A = 0$

$u(1) = 0 \Rightarrow \quad C\cancel{\ln 1}^{\,0} + D = 0 \Rightarrow D = 0$

$$u(x) = \begin{cases} B, & x<\xi \\ C\ln x, & x>\xi \end{cases}$$

The continuity of u at $x=\xi$:

$$\boxed{B = C\ln\xi}$$

The jump condition at $x=\xi$ $\quad (a_2(x) = -x)$:

$$a_2(\xi)\left[u'(\xi^+) - u'(\xi^-)\right] = 1$$

$$-\xi\left[\frac{C}{\xi}-0\right]=1: \quad \underline{\text{jump condition}}$$

$$u(x)=B, \quad x<\xi \quad \Rightarrow u'(x)=0 \Rightarrow u'(\xi^-)=0$$

$$u(x)=C\ln x, \quad x>\xi \quad \Rightarrow u'(x)=\frac{C}{x} \Rightarrow u'(\xi^+)=\frac{C}{\xi}$$

$$-C=1 \quad \Rightarrow \boxed{C=-1}$$

$$\therefore \ B=C\ln\xi=-\ln\xi \quad \Rightarrow \boxed{B=-\ln\xi}$$

Hence, the Green's function of (1)-(3) is

$$G(x,\xi)=\begin{cases}-\ln\xi, & 0\le x<\xi\\ -\ln x, & \xi<x\le 1\end{cases}$$

__Note__ again that $G(x,\xi)=G(\xi,x)$: symmetry

__Ex__ __An Initial Value Problem__

Consider IVP

(4) $$L(u)=f(t) \qquad 0\le t<\infty$$

prescribed

(5) $$u(0)=0$$

(6) $$u'(0)=0$$

Green's function satisfies

$$L(u) = \delta(t-\tau)$$

subject to homogeneous ICs:

$$u(0)=0, \quad u'(0)=0$$

Then solution of IVP (4)-(6) is given by superposition integral

$$u(t) = \int_0^t G(t,\tau)\, f(\tau)\, d\tau$$

Proof is similar to result for BVPs – HW

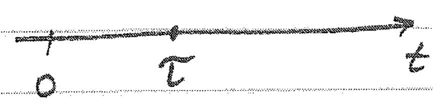
Special case:

$$L(u) = u'' + u$$

i.e.

(7) $$u'' + u = \delta(t-\tau)$$

(8) $$u(0) = u'(0) = 0$$



On $[0,\tau)$, $(\tau,+\infty)$, $\delta(t-\tau)=0 \Rightarrow u''+u=0$

$$u(t) = \begin{cases} C\sin t + \cancelto{0}{D}\cos t, & 0 \le t < \tau \\ A\sin t + B\cos t, & \tau < t < \infty \end{cases}$$

On $[0,\tau)$ we can apply ICs

$u(0)=0 \Rightarrow C\overset{0}{\cancel{\sin 0}} + D\overset{1}{\cos 0} = 0 \Rightarrow D=0$

$u'(0)=0, \quad u'(t) = C\cos t$

$C\underbrace{\cos 0}_{1} = 0 \Rightarrow C = 0$

$\Rightarrow u(t) \equiv 0 \quad \text{for} \quad t<\tau$

$$\therefore \quad u(t) = \begin{cases} 0, & t<\tau \\ A\sin t + B\cos t, & t>\tau \end{cases}$$

Continuity of $u(t)$ at $t=\tau$:

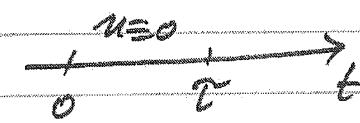
$$\boxed{0 = A\sin\tau + B\cos\tau} \quad \Rightarrow \quad B = -A\frac{\sin\tau}{\cos\tau}$$

The jump condition: $(a_2(t)=1)$:

$$a_2(\tau)\left[u'(\tau^+) - u'(\tau^-)\right] = 1$$

$t<\tau$: $u\equiv 0 \Rightarrow u'=0$

$\Rightarrow u'(\tau^-)=0$



$t>\tau$: $u(t) = A\sin t + B\cos t$

$u' = A\cos t - B\sin t$

$u'(\tau^+) = A\cos\tau - B\sin\tau$

$\therefore \quad 1\cdot\left[A\cos\tau - B\sin\tau - 0\right] = 1$ jump condition or $\boxed{A\cos\tau - B\sin\tau = 1}$

Recall
$$B = -A\frac{\sin\tau}{\cos\tau}$$

$$\Rightarrow \quad A\cos\tau + A\frac{\sin^2\tau}{\cos\tau} = 1$$

$$A\,\frac{\cos^2\tau + \sin^2\tau}{\cos\tau} = 1 \quad \Rightarrow \quad \boxed{A = \cos\tau}$$

$$\Rightarrow \boxed{B = -\sin\tau}$$

$$\therefore \; \underset{t>\tau}{u(t)} = A\sin t + B\cos t = \cos\tau\sin t - \sin\tau\cos t =$$

$$= \sin t\cos\tau - \cos t\sin\tau = \sin(t-\tau)$$

Hence, Green's function of IVP (7), (8) is

$$G(t,\tau) = \begin{cases} 0, & 0 \le t < \tau \\ \sin(t-\tau), & \tau < t < \infty \end{cases}$$

Then solution of IVP

$$u'' + u = f(t)$$
$$u(0) = u'(0) = 0$$

is
$$u(t) = \int_0^t \sin(t-\tau)f(\tau)\,d\tau$$