Impulses and Delta Functions

Consider a force acting on a short interval of time.

Ex Impulsive force of a bat striking a ball or a quick surge of voltage from a lightning bolt.

In such cases we may only need to know

$$p = \int_a^b f(t)\,dt \quad : \quad \text{impulse of force } f(t) \text{ over } [a, b]$$

Ex Particle of mass m with linear motion
Newton's 2^{nd} law

$$f(t) = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv)$$

a: acceleration; v: velocity
mv: momentum

Then

$$p = \int_a^b f(t)\,dt = \int_a^b \frac{d}{dt}(mv)\,dt =$$

$$= mv(b) - mv(a) : \text{ change of momentum}$$

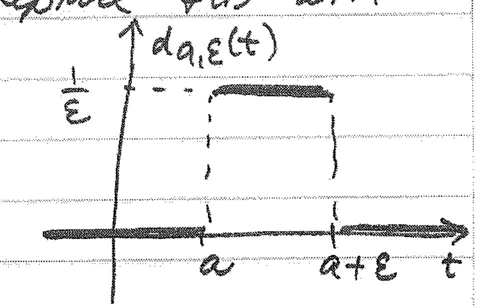
$\therefore$ impulse of force $=$ change of momentum

Hence, if we only need to know change of momentum, we need to know only the impulse of force and not the actual force or actual / precise time interval.

We would replace such a force $f(t)$ that acts on a very small time interval w/ a simple model that has the same impulse.

For simplicity, let $p = 1$. Replace $f(t)$ with

$$d_{a,\varepsilon}(t) = \begin{cases} \frac{1}{\varepsilon}, & a < t < a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$



Impulse of $d_{a,\varepsilon}(t)$ is

$$p = \int_a^b d_{a,\varepsilon}(t)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1 \quad \checkmark$$

$$b = a+\varepsilon$$

<u>Note</u>

$d_{a,\varepsilon}(t)$ has the same impulse for $\forall \varepsilon > 0$

$\Rightarrow$ we can write

$$\int_0^\infty d_{a,\varepsilon}(t)\,dt = 1$$

<u>Q</u> Can we think of instantaneous impulse at $t = a$?

Define

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t), \quad a \geq 0$$

<u>Dirac delta function</u>

Then

$$\int_0^\infty \delta_a(t)\,dt = \int_0^\infty \lim_{\varepsilon\to 0} d_{a,\varepsilon}(t)\,dt \underset{\substack{\int \text{ and lim} \\ \text{if we} \\ \text{can}}}{\overset{\text{swap}}{=}}$$

$$= \lim_{\varepsilon\to 0} \int_0^\infty d_{a,\varepsilon}(t)\,dt = \lim_{\varepsilon\to 0} 1 = 1$$

(1) $\Rightarrow \boxed{\int_0^\infty \delta_a(t)\,dt = 1}$

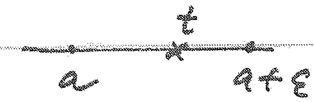
(2) But $\boxed{\delta_a(t) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}}$

No actual function can satisfy (1) and (2). But $\delta_a(t)$ is not an ordinary function, it is a generalized function or operator.

<u>Delta functions as operators</u>

If $g(t)$ is a continuous function, then by Mean Value Thm

$$\int_a^{a+\varepsilon} g(t)\,dt = \varepsilon\, g(\bar{t}), \quad \text{where} \quad \bar{t} \in [a, a+\varepsilon]$$



Then

$$\lim_{\varepsilon\to 0} \int_0^{\infty} g(t)\, d_{a,\varepsilon}(t) = \lim_{\varepsilon\to 0} \int_a^{a+\varepsilon} g(t)\cdot \frac{1}{\varepsilon}\, dt =$$

$$= \lim_{\varepsilon\to 0} \varepsilon\, g(\bar{t})\cdot \frac{1}{\varepsilon} = g(a)$$

If $\delta_a(t)$ were a function in a strict sense of def, and if we could interchange the order of lim and $\int$, we could get

$$\int_0^{\infty} g(t)\,\delta_a(t)\,dt = \int_0^{\infty} g(t) \lim_{\varepsilon\to 0} d_{a,\varepsilon}(t)\, dt \underset{\int \text{ and } \lim}{\overset{\text{swap}}{=}}$$

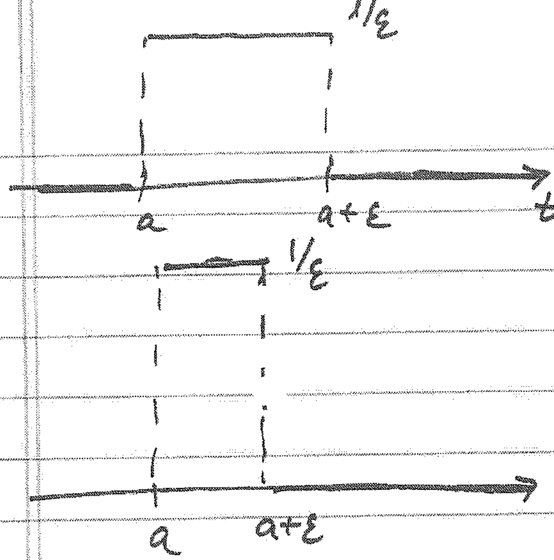
$$= \lim_{\varepsilon\to 0} \int_0^{\infty} g(t)\, d_{a,\varepsilon}(t)\, dt = g(a)$$

$$\Rightarrow \boxed{\int_0^{\infty} g(t)\,\delta_a(t)\,dt = g(a)}$$

we can take this as a def of symbol $\delta_a(t)$

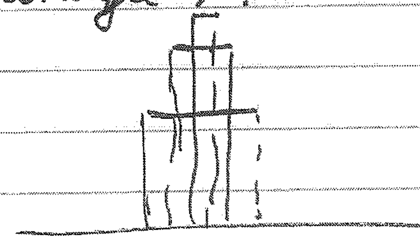
$\delta_a(t)$ specifies the operation $\int_0^{\infty} \ldots \delta_a(t)\,dt$

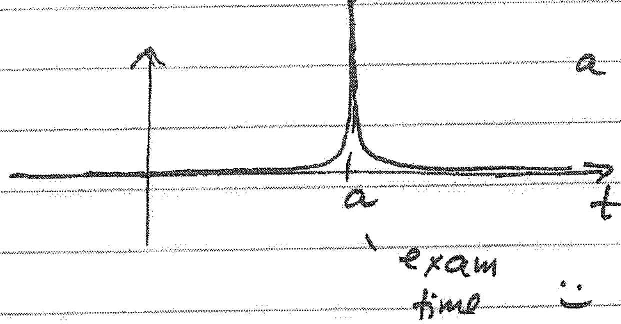
$$g(t) \xrightarrow{\delta_a(t)} g(a) \qquad \text{operator}$$



Piecewise constant approximation of $\delta_a(t)$

As $\varepsilon \to 0$, the height of rectangle $\nearrow$.





a continuous approximation of $\delta_a(t)$

Recall, Laplace transform of a function $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \quad , \quad s > 0$$

if $\int$ exists.

<u>Ex</u> $g(t) = e^{-st}$

$$\int_0^\infty e^{-st}\, \delta_a(t)\, dt = e^{-as}$$

$$\Rightarrow \quad \mathcal{L}\{\delta_a(t)\} = e^{-as}$$

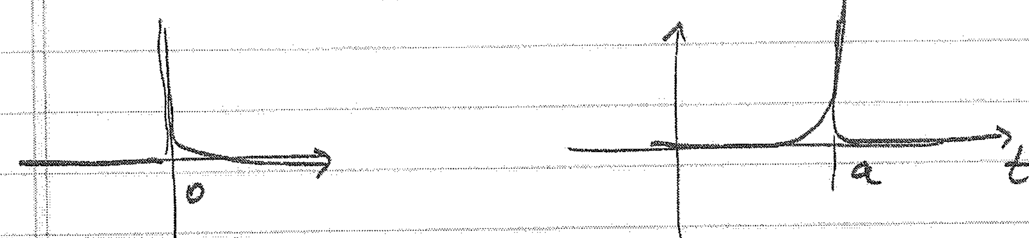
Thm

$$\mathcal{L}\{f(t)\} = F(s) \to 0 \quad \text{as} \quad s \to \infty$$

Note $\quad a = 0 \Rightarrow \quad \mathcal{L}\{\underbrace{\delta_0(t)}_{=\,\delta(t)}\} = \mathcal{L}\{\delta(t)\} = e^{-0 \cdot s} = 1 \not\to 0$

but $\delta(t)$ is not an ordinary function.

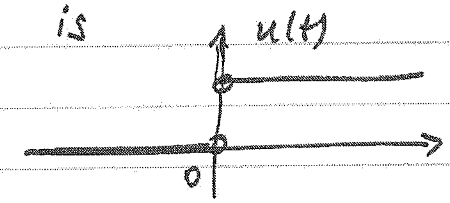
Notation $\quad \delta_a(t) = \delta(t-a)$



$\delta(t)$

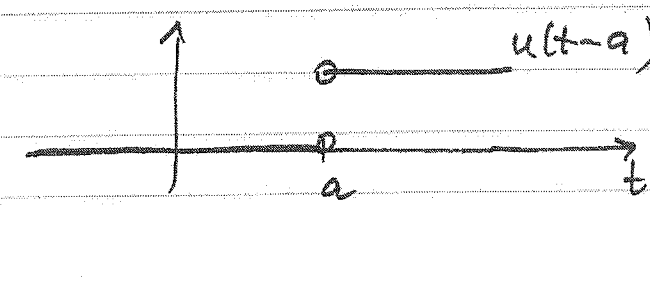
Delta Functions and Step Functions

The **unit step function** $u(t)$ is



$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

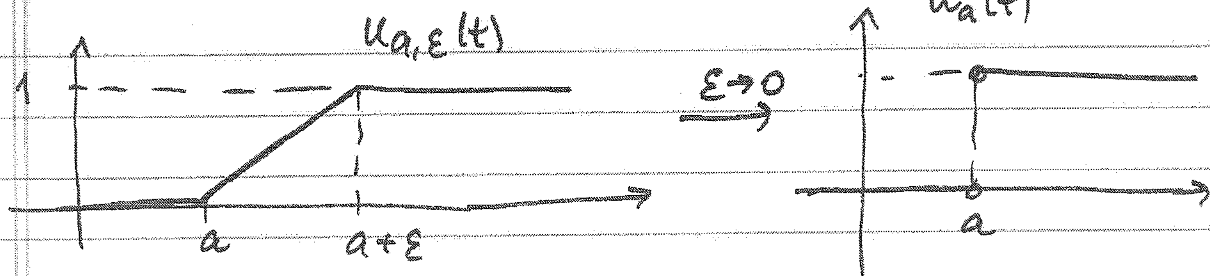
$$u(t-a) = \begin{cases} 0, & t-a < 0 \quad \text{or} \quad t < a \\ 1, & t-a > 0 \qquad\quad\;\; t > a \end{cases}$$



Another name of $u(t-a)$ is Heaviside function

Notation: $u(t-a) = u_a(t) = H(t-a)$.

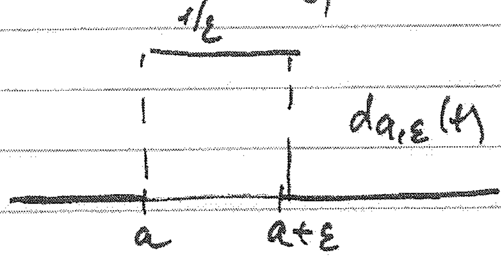
The value of $u(t-a)$ at $t=a$ can be defined as 0 or 1 or $\frac{1}{2}$ etc.



approximation of $u_a(t)$

$$\lim_{\varepsilon \to 0} u_{a,\varepsilon}(t) = u_a(t) = u(t-a)$$

$$\frac{d}{dt} u_{a,\varepsilon}(t) = \left\{ \begin{array}{ll} 0, & t < a \\ \frac{1}{\varepsilon}, & a < t < a+\varepsilon \\ 0, & t > a+\varepsilon \end{array} \right\} = d_{a,\varepsilon}(t)$$



Recall

$$\delta_a(t) = \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t)$$

Hence,

$$\frac{d}{dt} u_a(t) = \frac{d}{dt} \lim_{\varepsilon \to 0} u_{a,\varepsilon}(t) \overset{\text{swap}}{\underset{\frac{d}{dt} \text{ and } \lim}{=}} \lim_{\varepsilon \to 0} \frac{d}{dt} u_{a,\varepsilon}(t) =$$

$$= \lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \delta_a(t)$$

$$\Rightarrow \quad \boxed{\frac{d}{dt} u_a(t) = \delta_a(t) = \delta(t-a)}$$

formal def of derivative of unit step function

<u>Note</u> $u_a(t)$ is not differentiable in an ordinary sense at $t=a$

The Dirac Delta Function

$a \longrightarrow \xi$

Recall

$$\delta(x-\xi) = \begin{cases} 0, & x \neq \xi \\ \infty, & x = \xi \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-\xi)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(\xi)\, \delta(x-\xi)\, d\xi = f(x)$$

$$\delta(x-\xi) = \delta(\xi - x)$$

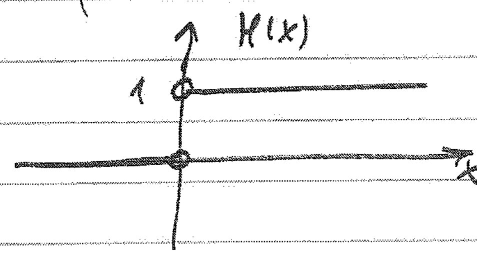
$$\int_{-\infty}^{\infty} \delta(x-\xi)\, d\xi = 1$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

$$u(x) = H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

unit step function

or Heaviside function



$$\delta(x-\xi) = \frac{d}{dx} H(x-\xi)$$

Properties of $\delta(x-\xi)$:

1. $\delta(x-\xi) = \delta(\xi - x)$: even function

2. $\delta(\underset{\text{const}}{\alpha}(x-\xi)) = \dfrac{1}{|\alpha|}\, \delta(x-\xi)$

3. derivative of delta function, $\delta'(x-\xi)$:

$$\int_{-\infty}^{\infty} f(\xi)\, \delta'(x-\xi)\, d\xi = -f'(x)$$