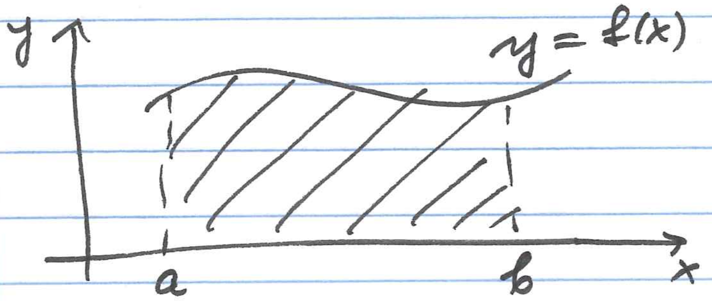
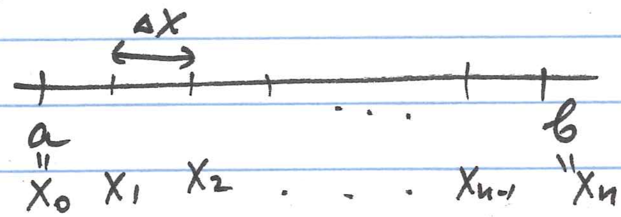
1/16/14

Riemann Sums



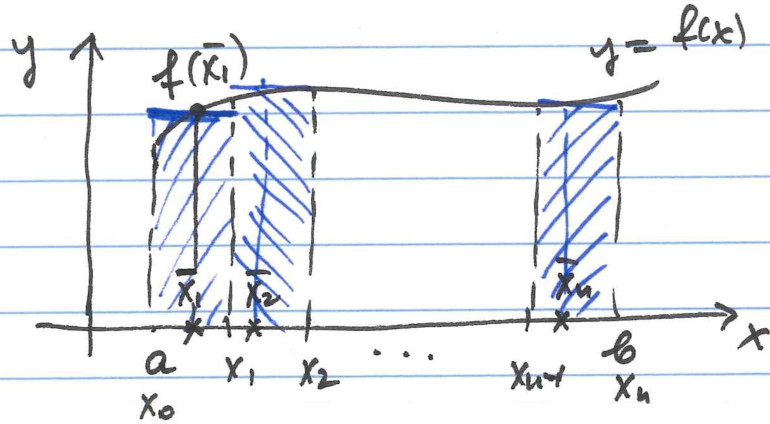
Regular partition:



$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$: grid points

$$\Delta x = \frac{b-a}{n} \;:\; \text{mesh size}$$

$$x_k = a + k\Delta x, \quad k = 0, 1, \dots, n$$



(1) $f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \dots + f(\bar{x}_n)\Delta x$: Riemann sum

$\bar{x}_k \in [x_{k-1}, x_k]$: $\bar{x}_k$ belongs to $[x_{k-1}, x_k]$

$\bar{x}_k = x_{k-1}$, left endpoint of $[x_{k-1}, x_k]$ $\Rightarrow$ (1) is called <u>left Riemann sum</u>

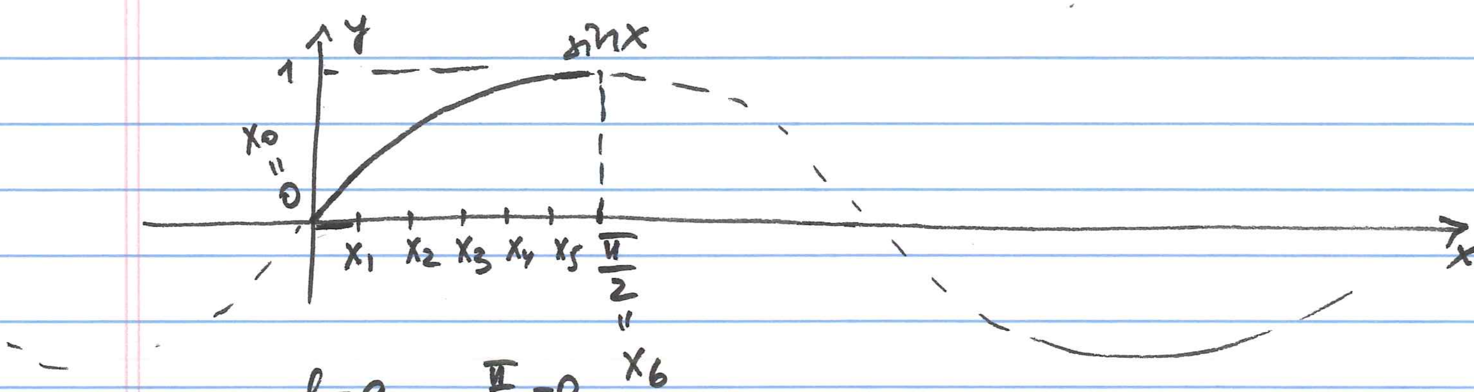
$\bar{x}_k = x_k$, right endpoint of $[x_{k-1}, x_k]$ $\Rightarrow$ (1) is called <u>right Riemann sum</u>

$\bar{x}_k = \dfrac{x_{k-1} + x_k}{2}$, midpoint of $[x_{k-1}, x_k]$ $\Rightarrow$ (1) is <u>midpoint Riemann sum</u>

$$f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \dots + f(\bar{x}_n)\Delta x =$$

$$= \sum_{k=1}^{n} f(\bar{x}_k)\Delta x$$

<u>Ex</u> Approximate area under $f(x) = \sin x$ and x-axis between $x=0$ and $x=\frac{\pi}{2}$ using a left Riemann sum with $n=6$.



$$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$x_0 = 0,\ x_1 = \frac{\pi}{12},\ x_2 = \frac{\pi}{6},\ x_3 = \frac{\pi}{4},\ x_4 = \frac{\pi}{3}$

$\bar{x}_1 = x_0 = 0$ $\qquad x_5 = \frac{5\pi}{12},\ x_6 = \frac{\pi}{2}$

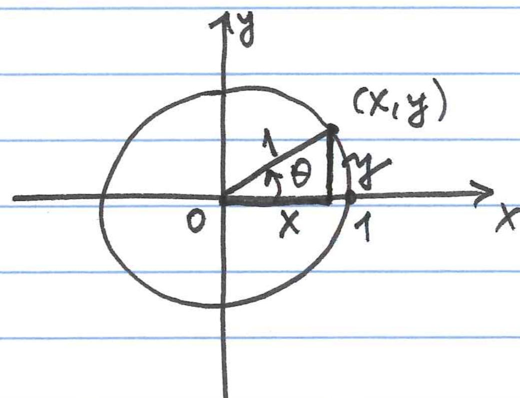
$\bar{x}_2 = \frac{\pi}{12}$

$\bar{x}_3 = \frac{\pi}{6},\ \bar{x}_4 = \frac{\pi}{4},\ \bar{x}_5 = \frac{\pi}{3},\ \bar{x}_6 = \frac{5\pi}{12}$

Left Riemann sum $= f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \dots +$

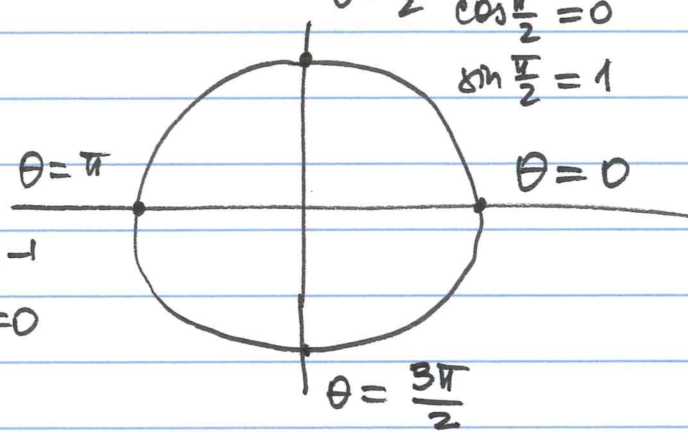
$+ f(\bar{x}_6)\Delta x = [\cancelto{0}{\sin(0)} + \sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right) +$

$+ \sin\left(\frac{\pi}{3}\right) + \sin\frac{5\pi}{12}]\,\frac{\pi}{12} \approx 0.863$



$\cos\theta = \frac{x}{1} = x$

$\sin\theta = \frac{y}{1} = y$



$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

$\cos 0 = 1$

$\sin 0 = 0$

$\cos\pi = -1$

$\sin\pi = 0$

$\cos\frac{3\pi}{2} = 0$

$\sin\frac{3\pi}{2} = -1$

$\boxed{\sin\frac{\pi}{6} = \frac{1}{2}}$ $\qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$

$$\tan\frac{\pi}{6} = \frac{\sin\pi/6}{\cos\pi/6} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$$\tan\frac{\pi}{3} = \sqrt{3}$$

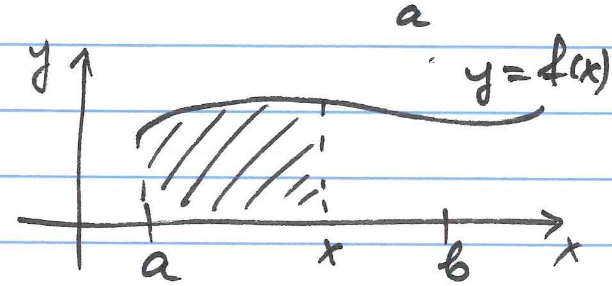
Definite integrals

$$\lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x} = \int_a^b f(x)\,dx$$ if $f(x)$ is integrable on $[a,b]$

Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a,b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b$$



is continuous on $[a,b]$ and differentiable on (a,b) and

$$A'(x) = f(x)$$

or $$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

i.e. $A(x)$ is an antiderivative of $f(x)$.

Let $F(x)$ be any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

$$= F(x)\Big|_{x=a}^{x=b}$$

Ex $$\int_0^{2\pi} 3\sin x\,dx = -3\cos x\Big|_0^{2\pi} = -3(\underbrace{\cos 2\pi}_{=1} - \underbrace{\cos 0}_{=1})$$

$$= -3(1-1) = 0$$

Differentiation of Integrals

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\int_a^{\varphi(x)} f(t)\,dt = \underbrace{f(\varphi(x))}_{f(t)\big|_{t=\varphi(x)}} \cdot \varphi'(x)$$

Ex $\frac{d}{dx}\int_0^{x^2} \cos t^2\, dt = \cos t^2 \Big|_{t=x^2} \cdot 2x =$

$= \cos x^4 \cdot 2x$

Substitution Rule

$$\int f(g(x))\, \underbrace{g'(x)\, dx}_{du} = \left| u = g(x) \right| = \int f(u)\, du$$

Ex $\int \cos^3 x \sin x\, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x \cdot dx \end{array} \right| =$

$= \int u^3 (-du) = -\frac{u^4}{4} + C =$

$= -\frac{\cos^4 x}{4} + C$

Ex $\int_0^2 \frac{dx}{(x+3)^3} = \left| \begin{array}{l} u = x+3 \\ du = dx \\ x=0 \Rightarrow u=3 \\ x=2 \Rightarrow u=5 \end{array} \right| = \int_3^5 \frac{du}{u^3} =$

$= \frac{1}{-3+1} u^{-3+1} \Big|_3^5 = -\frac{1}{2}\left(u^{-2}\right)\Big|_3^5 = -\frac{1}{2}\left(5^{-2} - 3^{-2}\right) =$

$= \frac{8}{225}$

As for general inverse functions that satisfy

$$f(f^{-1}(y)) = y \quad \text{and} \quad f^{-1}(f(x)) = x$$

we have

$$\sin(\sin^{-1}(x)) = x \quad \text{and} \quad \cos(\cos^{-1}(x)) = x$$

$$-1 \le x \le 1$$

$$\sin^{-1}(\sin(y)) = y \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\cos^{-1}(\cos(y)) = y \qquad 0 \le y \le \pi$$

1/17/2014

Inverse Functions (Review)

Consider a function $y=f(x)$. The function the reverses the action of f and maps $f(x)$ back to x is called <u>inverse function</u>

$$x \xrightarrow{f} \underset{\overset{\|}{y}}{f(x)} \xrightarrow{f^{-1}} x$$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & f(x)=y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

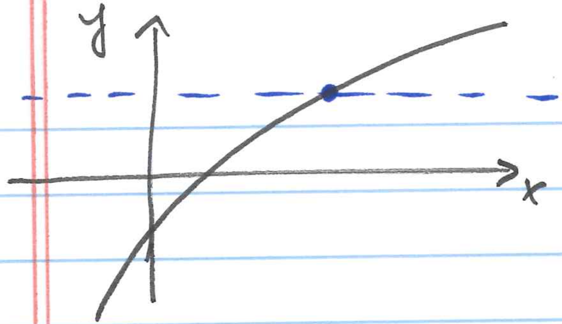
Note

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

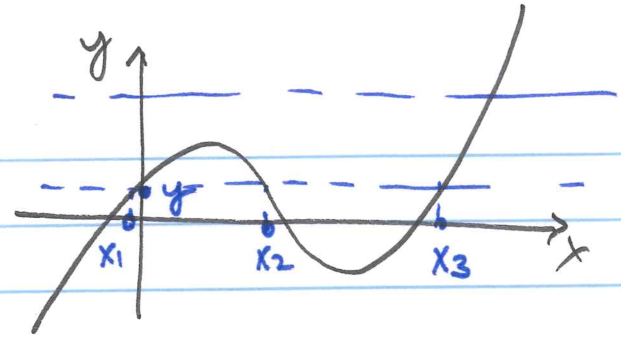
$$f^{-1}\circ f : \text{composition} \qquad f\circ f^{-1}$$

Def a function $f(x)$ is <u>one-to-one</u> if any value of $f(x)$ corresponds to exactly one value of x, ie. $f(x_1)\neq f(x_2)$ if $x_1\neq x_2$, or $f(x_1)=f(x_2) \Rightarrow x_1=x_2$

a <u>horizontal line test</u> checks if a function is one-to-one: any horizontal line can intersect a graph of a function at most once.



one-to-one



not one-to-one

If $f(x)$ is one-to-one function with domain D and range R, then this function has a unique inverse f^{-1} with domain R and range D, such that

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

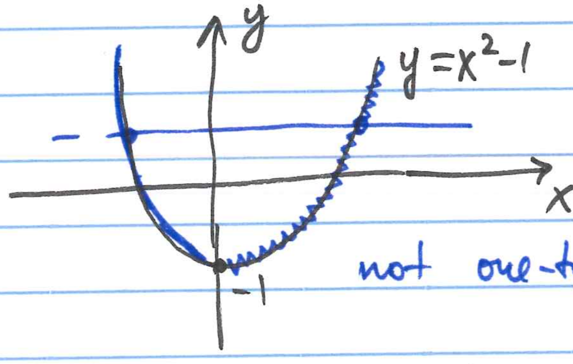
To find f^{-1}, you can swap x and y and then solve for y.

<u>Ex</u> $f(x) = 2x$ $\qquad y = 2x$

swap x and y $\Rightarrow$ $x = 2y$

solve for y $\Rightarrow$ $y = \frac{x}{2} = f^{-1}(x)$

<u>Ex</u> $f(x) = x^2 - 1$ $\Rightarrow$ $y = x^2 - 1$



not one-to-one for all x

$y = x^2 - 1$ is not invertible unless we restrict x to either $(-\infty, 0]$ or $[0, +\infty)$

Swap x and y : $x = y^2 - 1$

solve for y: $y^2 = x + 1$

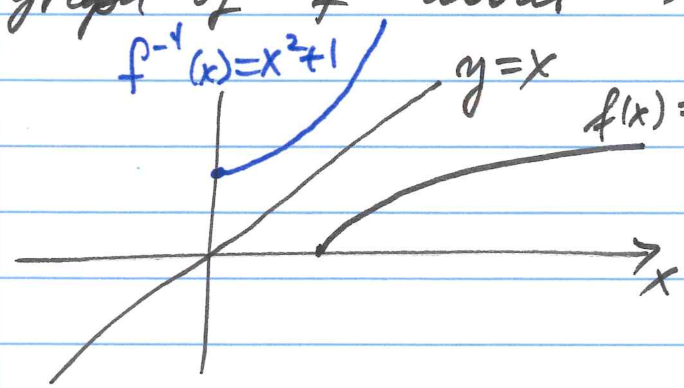
$y = \pm\sqrt{x+1}$

$y = +\sqrt{x+1}$ corresponds to $[0, +\infty)$

$y = -\sqrt{x^2+1}$ —"— $(-\infty, 0]$

Graphing the Inverse Function

To obtain graph of f^{-1} one can reflect graph of f about line $y = x$

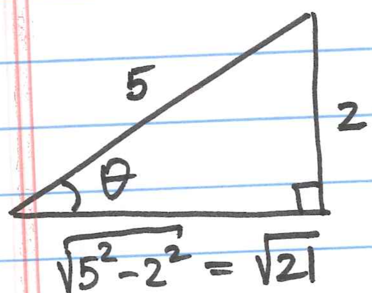


Graphs of f and f^{-1} are symmetric wrt $y = x$.

Right-triangle relationships

Ex Let $\theta = \sin^{-1}\left(\frac{2}{5}\right)$. Find $\cos\theta$ and $\tan\theta$

$$\sin\theta = \frac{2}{5}$$

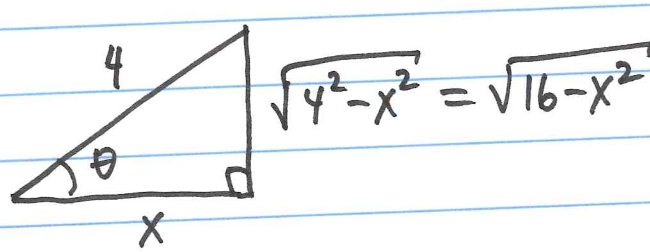


$$\Rightarrow \cos\theta = \frac{\sqrt{21}}{5}$$

$$\tan\theta = \frac{2}{\sqrt{21}}$$

Ex Express $\cot\left(\cos^{-1}\left(\frac{x}{4}\right)\right)$ in terms of x

Denote by $\theta = \cos^{-1}\left(\frac{x}{4}\right) \Rightarrow \cos\theta = \frac{x}{4}$



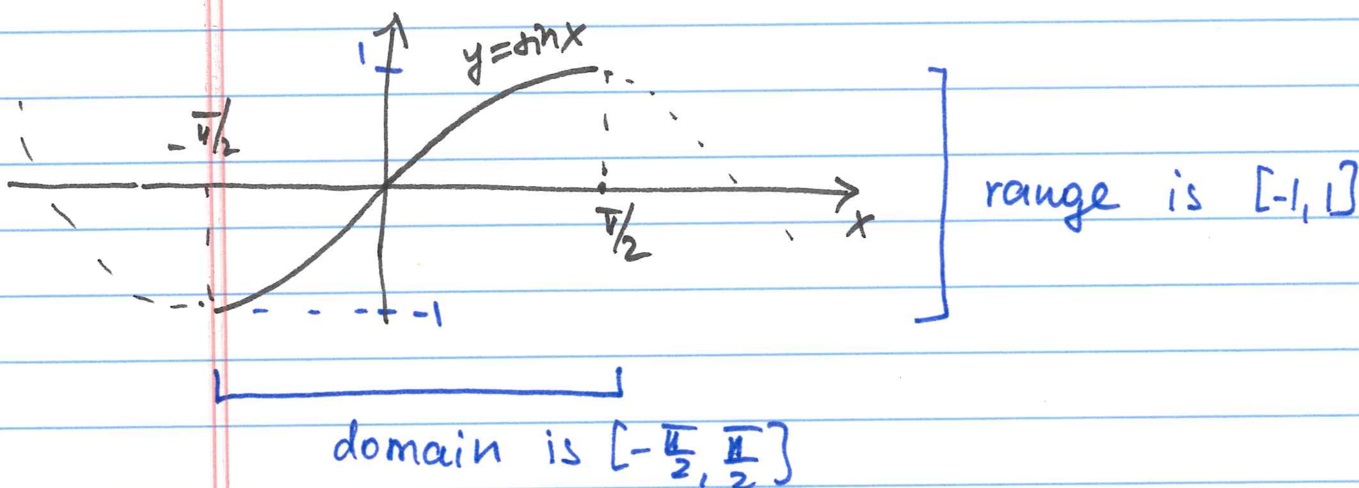
$$\sin\theta = \frac{\sqrt{16-x^2}}{4}$$

Hence, we need to find $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x/4}{\sqrt{16-x^2}/4} =$

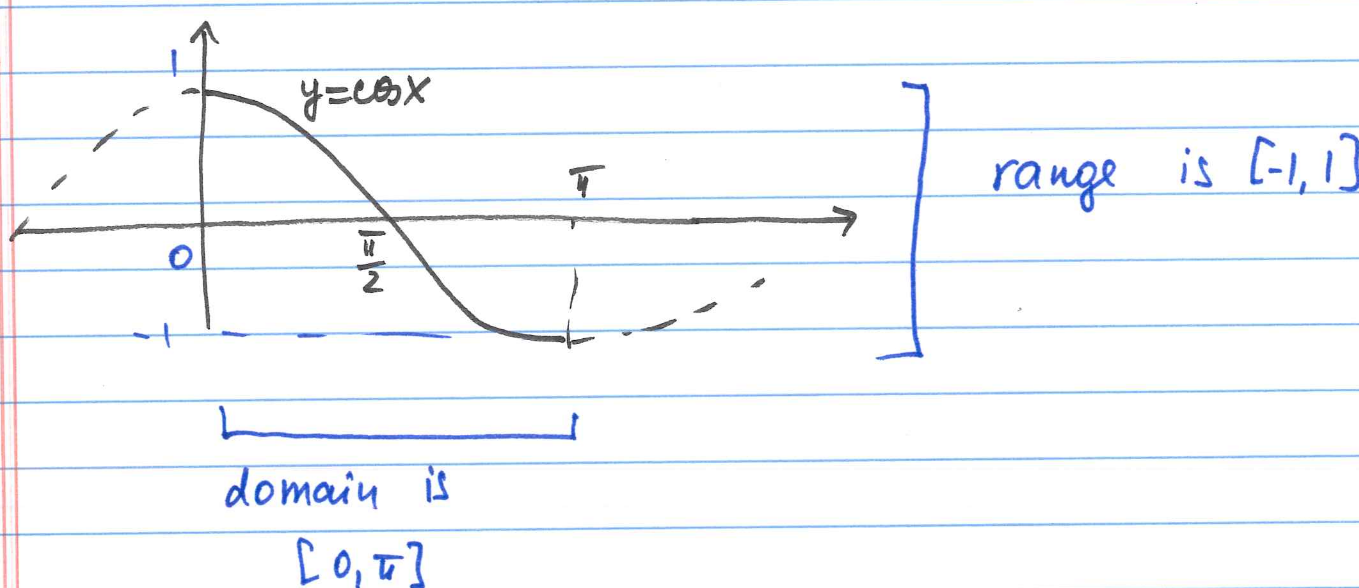
$$= \frac{x}{\sqrt{16-x^2}}$$

Since $y = \sin x$ is not one-to-one, we need to restrict domain in order to have the inverse function.

For $\sin x$, usually domain is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

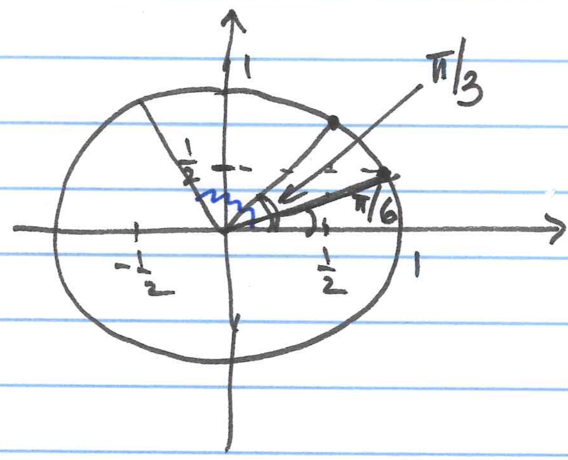


For cosine function, a standard choice is $[0, \pi]$



Q Find angle x on $[-\frac{\pi}{2}, \frac{\pi}{2}]$: $\sin x = \frac{1}{2} \Rightarrow \boxed{x = \frac{\pi}{6}}$

Q Find angle x on $[0, \pi]$: $\cos x = -\frac{1}{2}$



$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$

$\cos x = -\frac{1}{2} \Rightarrow x = \pi - \frac{\pi}{3}$

$= \boxed{\frac{2\pi}{3}}$

We define the inverse sine or arcsine, denoted by $y = \sin^{-1} x$ or $y = \arcsin x$, such that y is the angle : $\sin y = x$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$: range of $\sin^{-1} x$

Similarly we define inverse cosine or arccosine, denoted by $y = \cos^{-1} x$ or $y = \arccos x$ as y is the angle such that $\cos y = x$, $0 \le y \le \pi$: range of $\cos^{-1} x$

Domain for both $\sin^{-1} x$ and $\cos^{-1} x$ is $[-1, 1]$.

$-1 \le x \le 1$

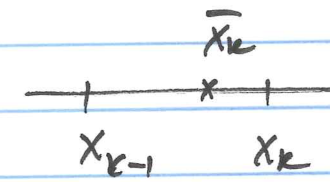
1/21/2014

$$\int_{-3}^{2} (1+x^2)\,dx \overset{?}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(\bar{x}_k)\,\Delta x =$$

$$f(x) = 1 + x^2$$

$$\Delta x = \frac{b-a}{n} \qquad a=-3,\ b=2$$

$$\Delta x = \frac{2-(-3)}{n} = \frac{5}{n}$$



$$= \lim_{n\to\infty} \sum_{k=1}^{n} (1+\bar{x}_k^{\,2}) \cdot \frac{5}{n}$$

$$\bar{x}_k \in [x_{k-1}, x_k]$$

$$x_k = a + k\cdot\Delta x = -3 + k\cdot\frac{5}{n}$$

Ex Find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Please correct this example.

Let $y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \sin y = \frac{\sqrt{3}}{2}$

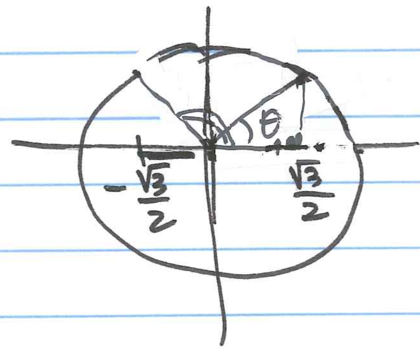
$$\Rightarrow \boxed{y = \frac{\pi}{\cancel{6}\,3}}$$

Note : $y = \frac{\pi}{\cancel{6}\,3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(in

Ex Find $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Denote $y=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow \cos y = -\frac{\sqrt{3}}{2}$



$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\cos(\underbrace{\pi - \frac{\pi}{6}}_{= y}) = -\frac{\sqrt{3}}{2}$

$\Rightarrow y = \frac{5\pi}{6}$

cos x is 2π-periodic

Ex $\cos^{-1}(\underset{\notin[0,\pi]}{\cos 3\pi}) =$

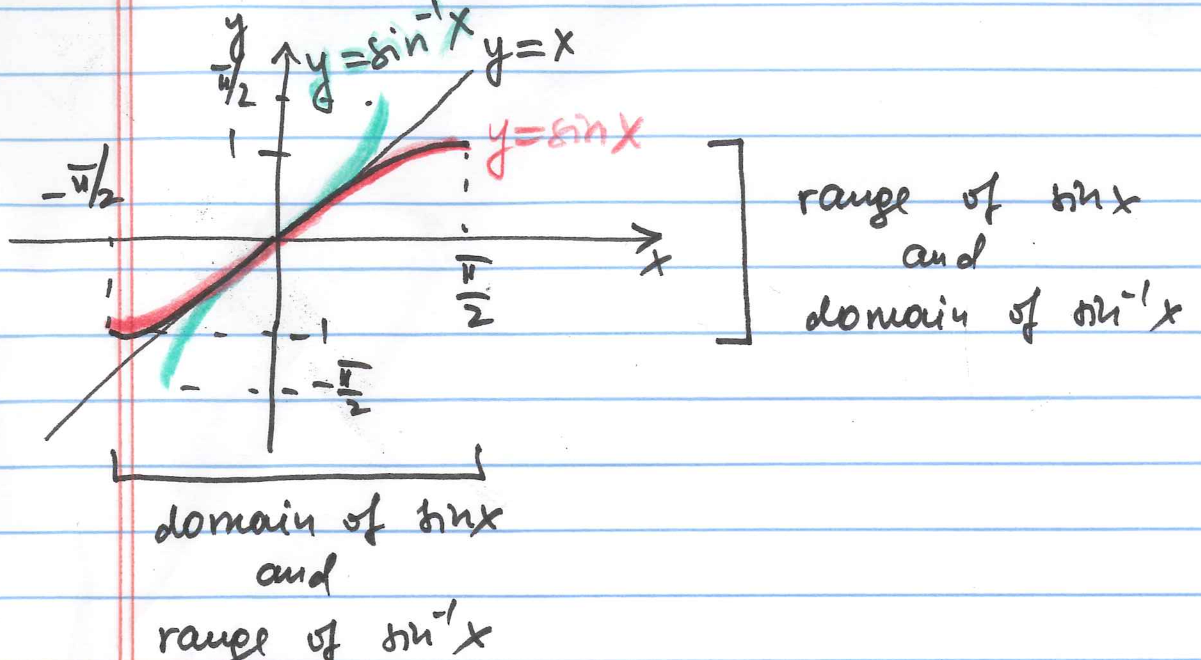
$\Rightarrow \cos(x + 2\pi \cdot n) = \cos x$

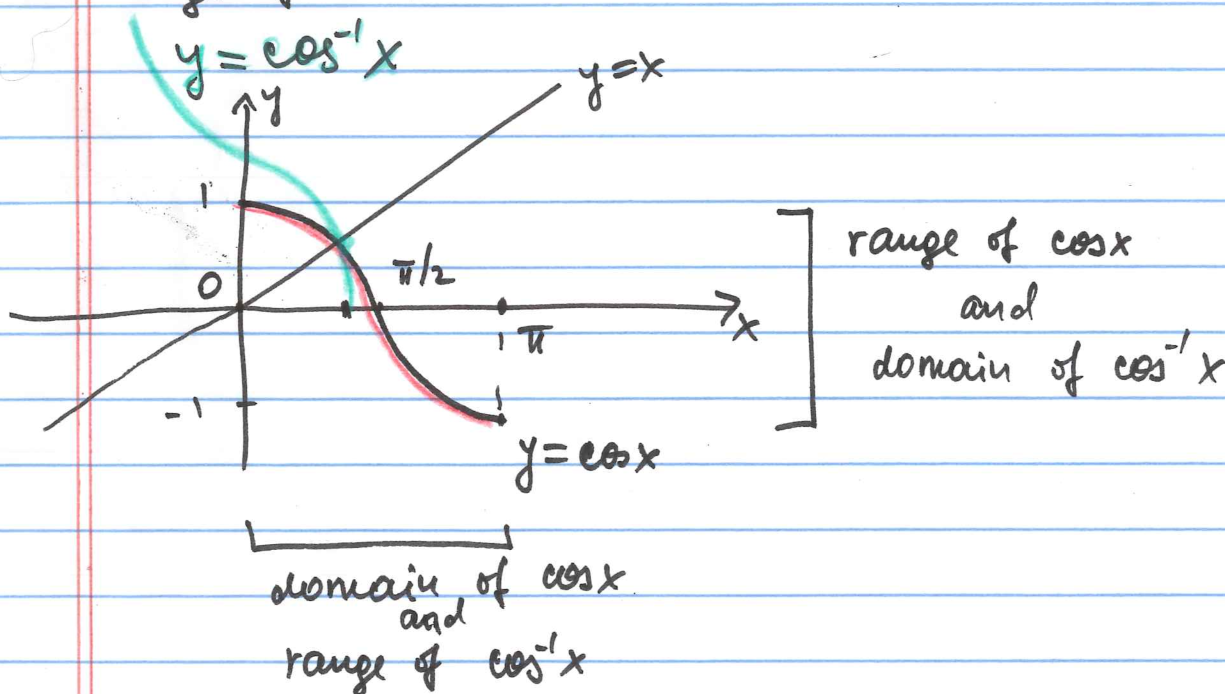
n is any integer

$= \cos^{-1}(\cos\underset{\in[0,\pi]}{[\pi+2\pi]}) = \cos^{-1}(\cos \underset{\in[0,\pi]}{\pi}) = \pi$

Ex $\sin(\sin^{-1}(\underset{\in[-1,1]}{\tfrac{1}{2}})) = \frac{1}{2}$

Graphs of $\sin^{-1}x$ and $\cos^{-1}x$





Derivative of inverse Function

$y = f(x)$

$y_0 = f(x_0)$

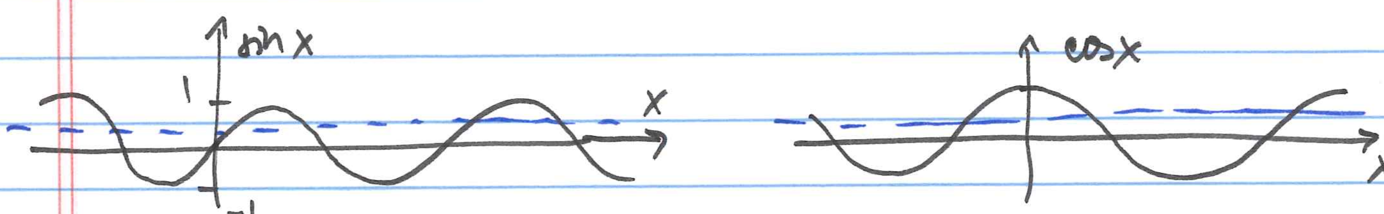
$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

7.5 Inverse Trigometric Functions

We know: given an angle x, we can find $\sin x$ or $\cos x$.

New: given a number y, find a value of angle x such that $\sin x = y$

similarly, given y, find angle x s.t. $\cos x = y$

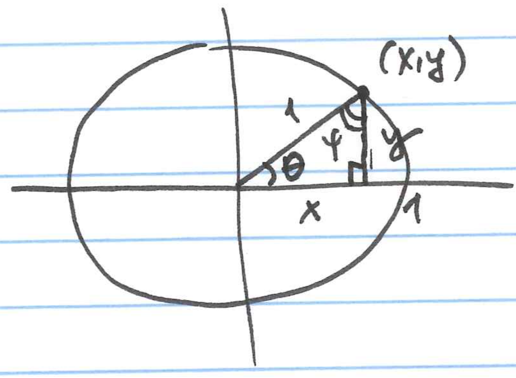


not one-to-one   not one-to-one

$|\sin x| \le 1 \Rightarrow |y| \le 1$

similarly $|\cos x| \le 1 \Rightarrow |y| \le 1$

Useful identity: $\boxed{\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}}$



$\theta + \varphi + \frac{\pi}{2} = \pi$

$\Rightarrow \boxed{\theta + \varphi = \frac{\pi}{2}}$

$\cos\theta = \frac{x}{1} = x \Rightarrow \boxed{\theta = \cos^{-1}x}$

$\sin\theta = y$

$\cos\varphi = y$

$\sin\varphi = x \Rightarrow \boxed{\varphi = \sin^{-1}x}$

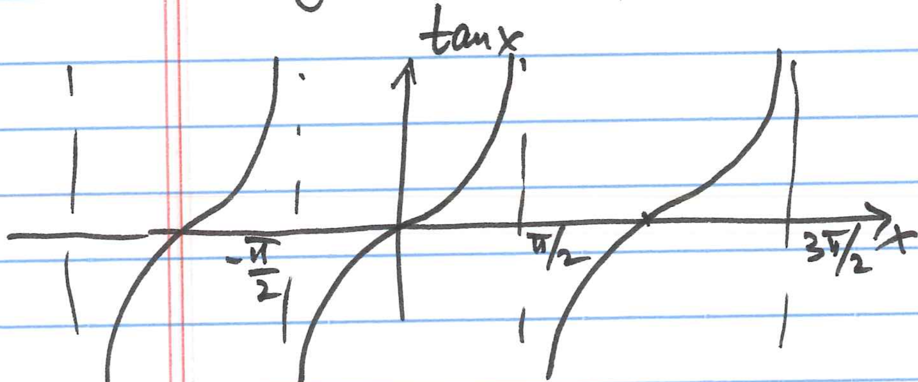
Hence $\theta + \varphi = \frac{\pi}{2}$ becomes $\boxed{\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}}$

$-1 \le x \le 1$

__Note__: angles θ, φ are called complementary angles

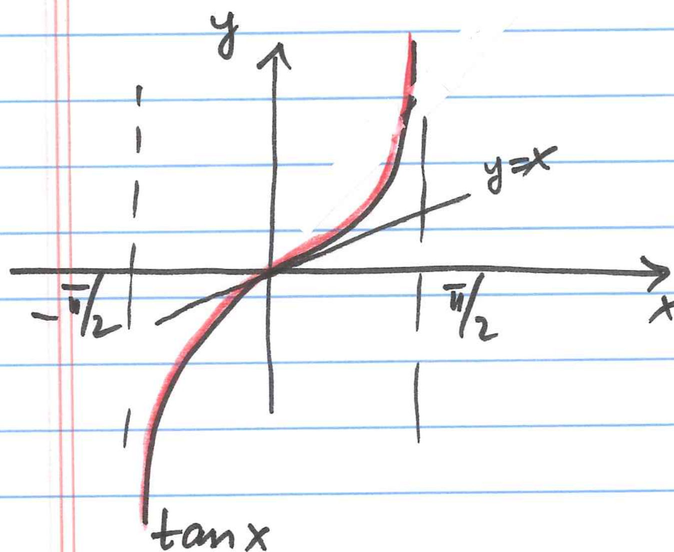
Other inverse trigonometric functions

Tangent function is one-to-one on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



and has inverse $y = \tan^{-1}x$ which is the value of y: $\tan y = x$

$(-\frac{\pi}{2}, \frac{\pi}{2})$: range of $\tan^{-1}x$



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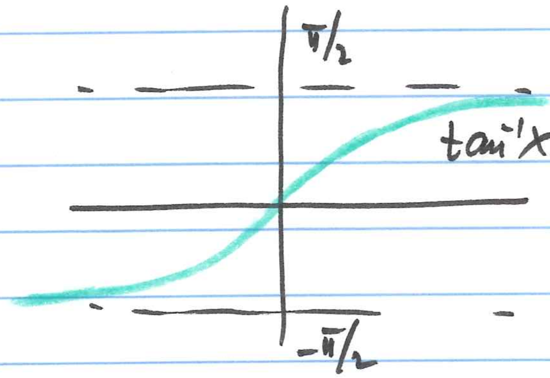


domain of $\tan^{-1}x$ is $(-\infty, +\infty)$

range of $\tan^{-1}x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$

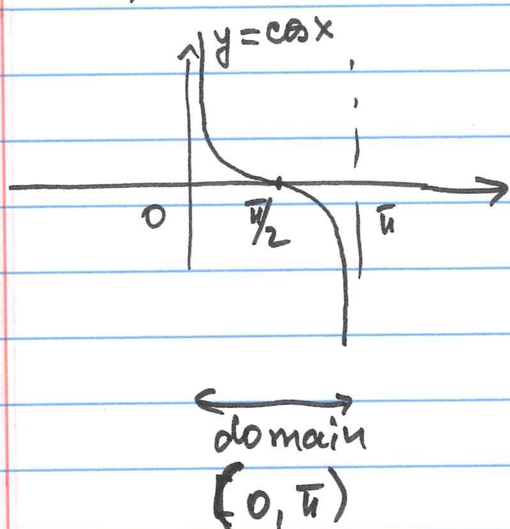
1/23/2014

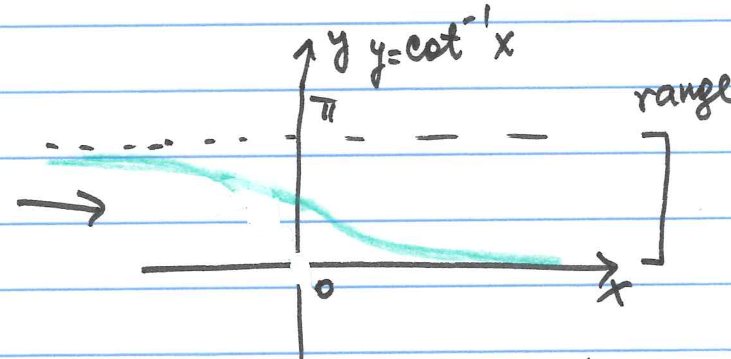
Other inverse trig. functions

* Cotangent is one-to-one on $(0,\pi)$ gives $y=\cot^{-1}x$ with range $(0,\pi)$

i.e. $\cot y = x$

$y=\cot x = \dfrac{\cos x}{\sin x}$





range is $(0,\pi)$

domain is $(-\infty,\infty)$

* Secant function is one-to-one on $[0,\pi]$ excluding $x=\frac{\pi}{2}$ gives $y=\sec^{-1}x$ with range $[0,\pi]$ excluding $x=\frac{\pi}{2}$

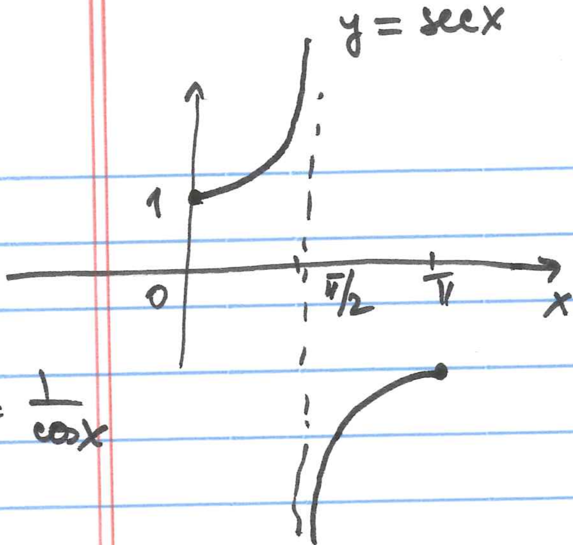
$y=\sec x = \dfrac{1}{\cos x}$

Notation: $[0,\pi] \setminus \{\frac{\pi}{2}\}$

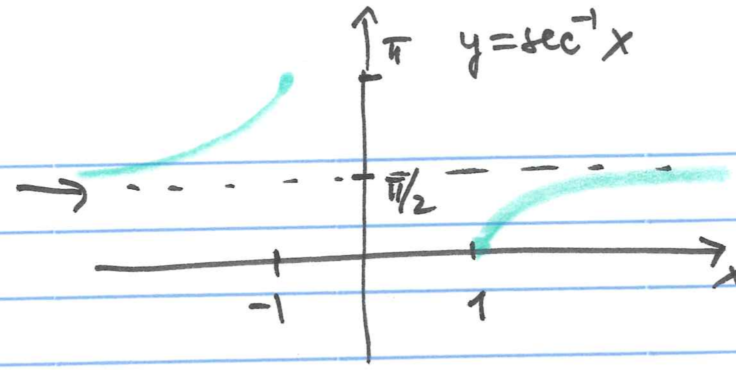
except

$y=\sec^{-1}x \Rightarrow \sec y = x$

$\sec x = \frac{1}{\cos x}$



domain is $[0,\pi] \setminus \{\frac{\pi}{2}\}$



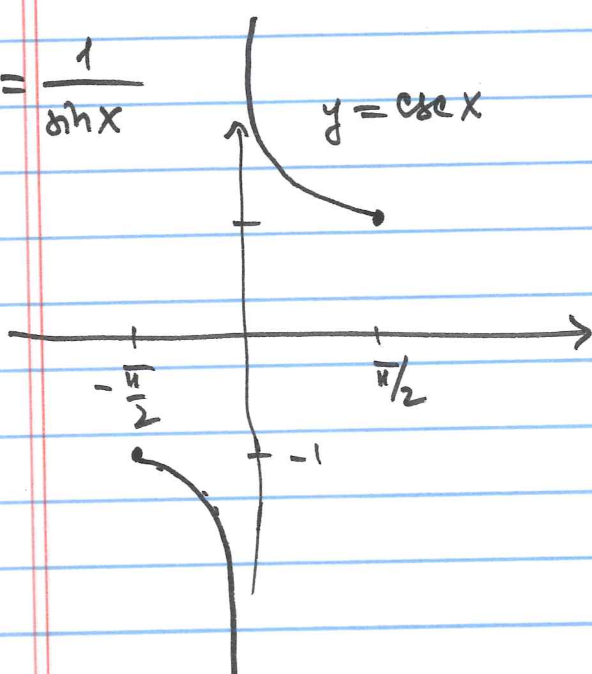
range is $[0,\pi] \setminus \{\frac{\pi}{2}\}$

domain is $|x| \geq 1$

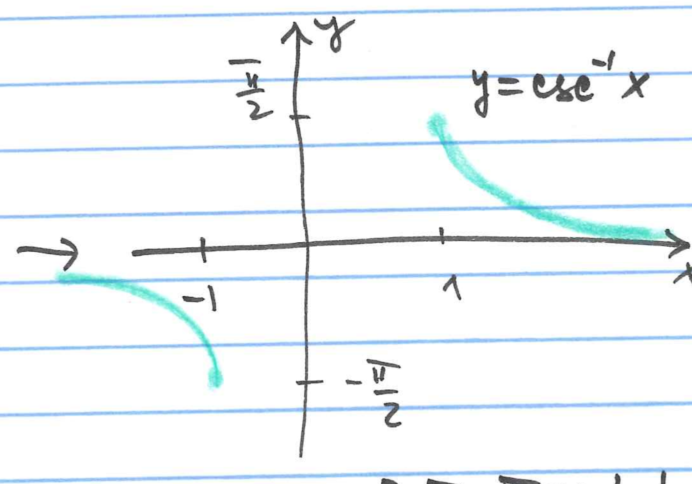
* Cosecant function is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$ gives $y = \csc^{-1}x$ such that $\csc y = x$.

$y = \csc^{-1}x$ has range $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

$y = \csc x = \frac{1}{\sin x}$



domain is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$



range is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

domain: $|x| \geq 1$

Ex Evaluate $\csc^{-1}(-1)$

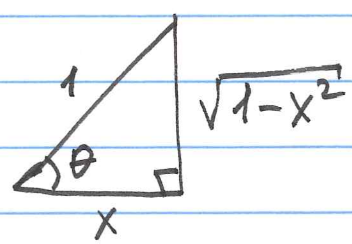
Let $y = \csc^{-1}(-1) \Rightarrow \csc y = -1$

$$\csc y = \frac{1}{\sin y}$$

$\Rightarrow \sin y = -1 \Rightarrow y = -\frac{\pi}{2}$. This value belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$ ✓

Ex Simplify $\tan(\cos^{-1} x)$

Let $\theta = \cos^{-1} x \Rightarrow \cos\theta = x = \frac{x}{1}$



$\therefore \tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$

Inverse sine and its derivative

Consider $y = \sin^{-1} x \Rightarrow \boxed{\sin y = x}$

Note: $y = y(x)$

Differentiate both sides of $\sin y = x$ wrt x

$\frac{d}{dx}$: $\sin y = x$

$$\cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

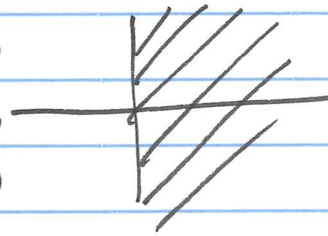
$$\cos^2 y + \sin^2 y = 1$$

$$\cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$$

$$y = \sin^{-1} x, \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

then $\cos y \geq 0$

for $y \in [\frac{\pi}{2}, \frac{\pi}{2}]$



$\therefore \quad \cos y = \sqrt{1-x^2}$ (with "+" sign)

Hence,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

or

$$\boxed{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1}$$

Other derivatives:

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}, \qquad -1 < x < 1$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$-\infty < x < \infty;$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$|x|>1$$

Ex Evaluate $f'(x)$ if $f(x) = \sin^{-1}(e^{-2x})$

$$f'(x) = \frac{1}{\sqrt{1-(e^{-2x})^2}} \cdot \underbrace{e^{-2x} \cdot (-2)}_{\frac{d}{dx}(e^{-2x})}$$

Integrals involving inverse trigonometric functions

Recall

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \qquad |x|<1$$

Chain Rule:

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \underbrace{\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}}_{} \cdot \underbrace{\frac{1}{a}}_{\frac{d}{dx}\left(\frac{x}{a}\right)} = \frac{1}{\sqrt{a^2-x^2}}$$

$a>0$

$$\left|\frac{x}{a}\right|<1$$

or $|x|<a$

$$\sqrt{1-\frac{x^2}{a^2}} = \sqrt{\frac{a^2-x^2}{a^2}} = \frac{\sqrt{a^2-x^2}}{a}$$

$$\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} = \frac{a}{\sqrt{a^2-x^2}}$$

Hence

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C, \qquad a>0$$

Similarly,

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C, \qquad a\neq 0$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, \qquad a>0$$

Ex Evaluate $\displaystyle\int \frac{2}{16z^2+25}\,dz = \frac{1}{16}\int \frac{2}{z^2+\frac{25}{16}}\,dz =$

$a = \frac{5}{4}$

$$= \frac{1}{16}\cdot\frac{1}{\frac{5}{4}}\tan^{-1}\left(\frac{z}{\frac{5}{4}}\right) + C =$$

$$= \frac{1}{16}\cdot\frac{4}{5}\tan^{-1}\left(z\cdot\frac{4}{5}\right) + C = \frac{1}{20}\tan^{-1}\left(\frac{4z}{5}\right) + C$$

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4.7 L'Hôpital's Rule

Recall, a function $f(x)$ is <u>continuous</u> at $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

This means that we can evaluate $\lim_{x\to a}$ by substitution of $x=a$ into $f(x)$.

<u>Ex</u> $\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$ but $\lim_{x\to 0} \frac{\sin x}{x} = 1$

we cannot evaluate limit here using substitution

When we have situation $\frac{0}{0}$ or $\frac{\infty}{\infty}$ we say that this limit is <u>undeterminate form</u>

<u>Ex</u> $\lim_{x\to\infty} \frac{ax}{x} = \frac{\infty}{\infty}$

$a \neq 0$ $\lim_{x\to\infty} \frac{ax}{x} = \lim_{x\to\infty} a = a$: <u>finite #</u> $\neq 0$

<u>Ex</u> $\lim_{x\to\infty} \frac{x^2}{x} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} \frac{x}{1} = \infty$: <u>infinite limit</u>

ex $\lim_{x\to\infty} \frac{x}{x^2} = \frac{\infty}{\infty}$

but $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$: zero limit

L'Hôpital's Rule for the Form $\frac{0}{0}$

Consider $f(x)$, $g(x)$ and assume

$\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$: undeterminate limit

Thm L'Hôpital's Rule

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on RHS exists (or is $\pm\infty$). The rule also applies if $x\to a$ is replaced by $x\to\pm\infty$, $x\to a^+$, $x\to a^-$.

Proof (special case)

Assume f', g' are continuous at $x=a$, $f(a)=g(a)=0$, and $g'(a)\neq 0$.

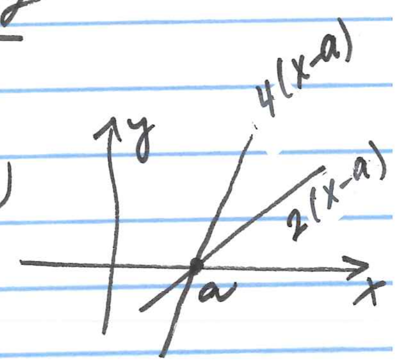
$$\lim_{x\to a}\frac{f'(x)}{g'(x)} \underset{\substack{\text{are}\\ \text{continuous}}}{\overset{f',\,g'}{=}} \frac{f'(a)}{g'(a)} \underset{\text{derivative}}{\overset{\text{def of}}{=}}$$

$$= \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} \underset{\substack{\text{of}\\ \text{quotient}}}{\overset{\lim}{=}} \lim_{x\to a}\frac{\frac{f(x)-f(a)}{\cancel{x-a}}}{\frac{g(x)-g(a)}{\cancel{x-a}}} =$$

$$= \lim_{x\to a}\frac{f(x)-\overset{\to 0}{f(a)}}{g(x)-\underset{\to 0}{g(a)}} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

$$\therefore \quad \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f(x)}{g(x)} \quad \blacksquare$$

Geometric interpretation of l'Hôpital's rule



Let $f(x)=4(x-a)$, $g(x)=2(x-a)$

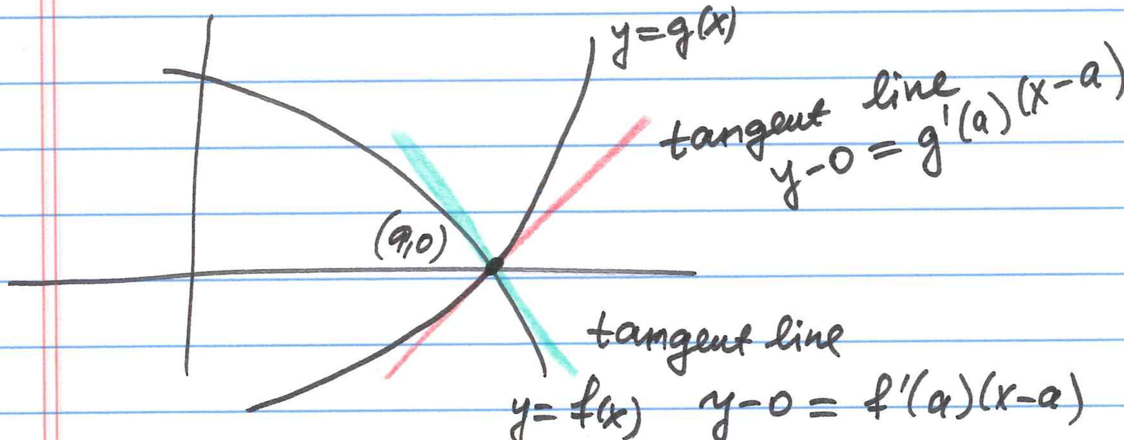
$\Rightarrow f(a)=g(a)=0$

$f'(x)=f'(a)=4$, $\underset{=g'(x)}{g'(a)}=2$: slopes of f & g

$$\therefore \quad \frac{f(x)}{g(x)} = \frac{4(x-a)}{2(x-a)} = \frac{4}{2} = \frac{f'(x)}{g'(x)}$$

we have exact equality: $\dfrac{f(x)}{g(x)} = \dfrac{f'(x)}{g'(x)}$

If $f(x)$ and $g(x)$ are nonlinear, we can use their linearization (linear approximation) at $(a,0)$.



<u>line</u> through pt (x_0, y_0) with slope k:

$$y - y_0 = k(x - x_0)$$

<u>tangent line</u> : $y - y_0 = \underset{\frac{dy}{dx}(x_0)}{\underset{\parallel}{f'(x_0)}}(x - x_0)$

$(a,0) \Rightarrow x_0 = a,\ y_0 = 0$

near $x \approx a$

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

$$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$\underline{\underline{Ex}}$ $$\lim_{x\to 0} \frac{1-\cos 3x}{8x^2} = \frac{0}{0} \overset{\text{l'Hôp.}}{\underset{\text{rule}}{=}} \lim_{x\to 0} \frac{\sin 3x \cdot 3}{16x} =$$

$$= \frac{0}{0} \overset{\text{l'Hôp.}}{\underset{\text{rule}}{=}} \lim_{x\to 0} \frac{9\cos 3x}{16} = \frac{9}{16}$$

<u>Undeterminate form $\frac{\infty}{\infty}$</u>

$\underline{\underline{Ex}}$ $$\lim_{x\to\infty} \frac{3x^4-x^2}{6x^4+12} = \frac{\infty}{\infty} \overset{\text{l'Hôp.}}{\underset{\text{rule}}{=}} \lim_{x\to\infty} \frac{12x^3-2x}{24x^3} =$$

$$= \frac{\infty}{\infty} \overset{\text{l'Hop.}}{\underset{\text{rule}}{=}} \lim_{x\to\infty} \frac{36x^2-2}{24.3x^2} = \frac{\infty}{\infty} \overset{\text{l'Hop.}}{\underset{\text{rule}}{=}}$$

$$= \lim_{x\to\infty} \frac{72x}{72\cdot 2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

Related Underterminate Forms

$0 \cdot \infty$ and $\infty - \infty$

Ex

$$\lim_{x\to 0} x \cdot \underbrace{\csc x}_{= \frac{1}{\sin x}} = 0 \cdot \infty = \lim_{x\to 0} \underbrace{x \cdot \frac{1}{\sin x}}_{0 \,\cdot\, \infty} = \lim_{x\to 0} \frac{x}{\sin x} =$$

$$= \frac{0}{0} \overset{H}{=}$$

Now we can use l'Hôpital's Rule!

$$\overset{H}{=} \lim_{x\to 0} \frac{1}{\cos x} = 1$$

$$x \cdot \csc x = \frac{\csc x}{\frac{1}{x}} = \frac{\infty}{\infty}$$: not the best approach in this particular example

Ex

$$\lim_{x\to\infty} (x - \underbrace{\sqrt{x^2+1}}_{\approx \sqrt{x^2} = x}) = \infty - \infty$$

we use conjugate factor approach

$$\lim_{x\to\infty} (x - \sqrt{x^2+1}) = \lim_{x\to\infty} \frac{(x - \sqrt{x^2+1})(x+\sqrt{x^2+1})}{x+\sqrt{x^2+1}} \overset{}{=}$$

Recall

$$(a-b)(a+b) = a^2 - b^2$$

$$\underbrace{\left(x-\sqrt{x^2+1}\right)}_{(a\,-\,b)}\underbrace{\left(x+\sqrt{x^2+1}\right)}_{(a\,+\,b)} = \underbrace{x^2 - \left(\sqrt{x^2+1}\right)^2}_{a^2\,-\,b^2} =$$

$$= x^2 - (x^2+1) = -1$$

$$= \lim_{x\to\infty} \frac{x^2 - (x^2+1)}{x+\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{-1}{x+\sqrt{x^2+1}} = \boxed{0}$$

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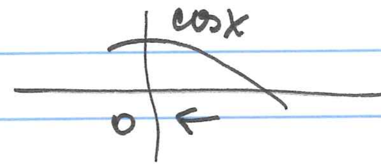
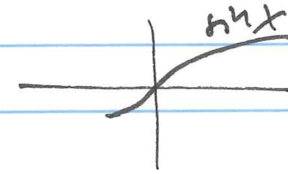
Ex

$$\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \infty - \infty$$

$$\cot x = \frac{\cos x}{\sin x}$$





$$\lim_{x\to 0^+}\cot x = \frac{1}{0^+} = +\infty$$

Method I

Recall $\lim_{x\to 0}\frac{\sin x}{x} = 1 \Rightarrow \sin x \sim x$ as $x\to 0$

$$\Rightarrow \lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) \underset{\sin x\sim x}{=}$$

$$= \lim_{x\to 0^+}\left(\frac{\cos x}{x} - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{\cos x - 1}{x} = \frac{0}{0} \underset{\text{rule}}{\overset{\text{l'Hôpital}}{=}}$$

$$= \lim_{x\to 0^+}\frac{-\sin x}{1} = 0$$

Method II

$$\lim_{x\to 0^+}\left(\cot x - \frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x\cos x - \sin x}{x\sin x}$$

$$= \frac{0}{0} \underset{\text{rule}}{\overset{\text{l'Hop.}}{=}} \lim_{x\to 0^+}\frac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{\sin x + x\cos x} = \frac{0}{0} \underset{\text{rule}}{\overset{\text{l'Hop.}}{=}}$$

$$\left(u(x)v(x)\right)' = u'(x)v(x) + u(x)v'(x)$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{\left(v(x)\right)^2}$$

$$= \lim_{x\to 0^+} \frac{\overset{\nearrow 0}{-\sin x} - \overset{\nearrow 0}{x\cos x}}{\underset{\downarrow\atop 1}{\cos x} + \underset{\downarrow\atop 1}{\cos x} - \underset{\downarrow\atop 0}{x\sin x}} = \frac{0}{2} = 0$$

7.6 L'Hôpital's Rule and Growth Rates of Functions

We will consider other indeterminate forms: 1^∞, 0^0, ∞^0. These form arise from the limit:

$$\lim_{x\to a} f(x)^{g(x)}$$

Recall $e^{\ln x} = x$, $\ln x^{b} = b\ln x$

Note: $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$

Then
$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} =$$

exp function is continuous $\Rightarrow$

$$= e^{\boxed{\lim_{x\to a} g(x)\ln f(x)}}$$

$$\lim_{t\to a} \exp(t) = \exp\left(\lim_{t\to a} t\right)$$

<u>Aside</u>

$h(x)$ is continuous at $x=a$

$$\Rightarrow \lim_{x\to a} h(x) = h(a) =$$

$$= h\left(\lim_{x\to a} x\right)$$

Now can consider

$$\lim_{x\to a} g(x)\ln f(x)$$

If this limit exists, we can evaluate

$\lim_{x\to a} f(x)^{g(x)}$ in two steps.

<u>Step 1</u> Evaluate $L = \lim_{x\to a} g(x)\ln f(x)$

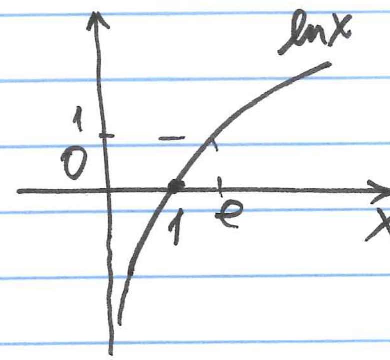
This limit can usually be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and computed using l'Hopital Rule.

<u>Step 2</u> Then $\lim_{x\to a} f(x)^{g(x)} = e^L$.

Ex Evaluate $\lim_{x\to0^+} x^x = 0^0$

$$x^x = e^{\ln x^x} = e^{x\ln x}$$

$$\Rightarrow \lim_{x\to0^+} x^x = \lim_{x\to0^+} e^{x\ln x} = e^{\lim_{x\to0^+} x\ln x} \quad = L$$



Compute $L = \lim_{x\to0^+} x\ln x = 0\cdot(-\infty)$

$$= \lim_{x\to0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty} \underset{\text{Rule}}{\overset{\text{l'Hôp.}}{=}}$$

$\ln 1 = 0$

$\ln e = 1$

$$= \lim_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+} \frac{1}{x}(-x^2)$$

$$= \lim_{x\to0^+}(-x) = 0 \quad \Rightarrow \quad L=0$$

$$\therefore \lim_{x\to0^+} x^x = e^L = e^0 = \boxed{1}$$

Ex $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} =$$

$$= \lim_{x\to\infty} e^{x \ln(1+\frac{1}{x})} = e^{\lim_{x\to\infty} x \ln(1+\frac{1}{x})}$$

$$L = \lim_{x\to\infty} x \ln(1+\frac{1}{x}) = \infty \cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \frac{0}{0} \underset{\text{rule}}{\overset{\text{l'Hop.}}{=}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{(-\frac{1}{x^2})}}{\cancel{-\frac{1}{x^2}}} = 1 \Rightarrow L = 1$$

$$\therefore \lim_{x\to\infty} (1+\frac{1}{x})^x = e^L = e^1 = e.$$

Growth Rates of Functions

L'Hôpital's Rule allows one to compare growth rates of functions.

Ex $N(t)$: # of infected people

$$N(t) = 2.5t^2 e^{-0.01t} = 2.5 \frac{t^2}{e^{0.01t}} = \frac{\infty}{\infty}$$

$$\infty \cdot 0$$

Q as $t \to \infty$, does the epidemic spread

$(N(t) \to \infty)$ or dies out $(N(t) \to 0)$.

$N(t) \to \infty$ means that $\frac{t^2}{e^{0.01t}} \to \infty$

i.e. t^2 grows faster than $e^{0.01t}$

$N(t) \to 0$ means that $\frac{t^2}{e^{0.01t}} \to 0$

i.e. $e^{0.01t}$ grows faster than t^2.

We will get ranking of the following functions:

- mx, $m>0$: linear functions
- x^p, $p>0$: polynomial or algebraic functions
- x^x : superexponential or tower function
- $\ln x$: logarithmic functions
- $\ln^q x$, $q>0$: powers of logarithmic functions
- $x^p \ln x$, $p>0$: combination of powers and logarithmic function
- e^x: exponential functions.

Suppose $f(x)$ and $g(x)$ grow at ∞, i.e.
$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = \infty$

We say that <u>f grows faster than g</u> if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \quad \text{or} \quad \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$$

The function f and g have <u>comparable growth rate</u> if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M = \text{const} \; : \; \text{finite } \#, \neq 0$$
$$0 < M < \infty$$

Ex $\lim\limits_{x\to\infty} \frac{x^2+1}{x^2} = 1$ $\quad \underset{\text{rule}}{\text{l'Hop.}} \lim\limits_{x\to\infty} \frac{2x}{2x} = 1$

$x^2+1 \sim x^2$

x^2+1 and x^2 have the same growth rate

$$\lim_{x\to\infty} \frac{x^2+1}{x^2} \underset{\text{Rule}}{\overset{\text{l'Hop.}}{=}} \lim_{x\to\infty} \frac{2x}{2x} = 1$$

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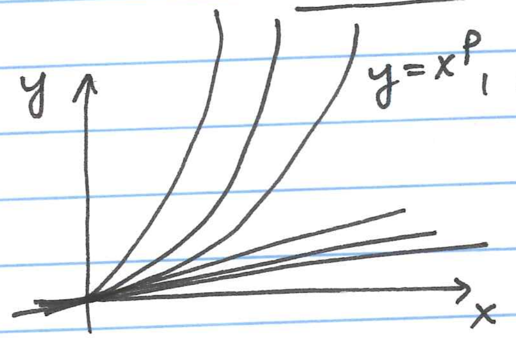
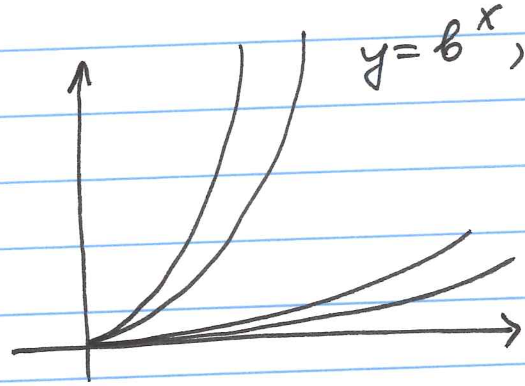
7.6 Growth Rates (Cont'd)



$y = x^p,\ p > 1$: polynomial functions

$y = mx$: linear functions

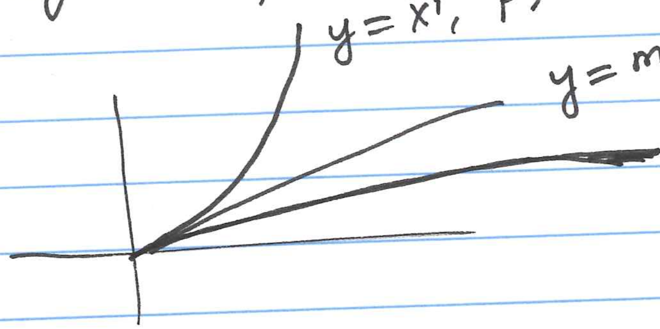
$\therefore$ any polynomial function grows faster than any linear function, or polynomials have greater growth rate



$y = b^x,\ b > 1$: exponential functions

$y = x^p,\ p > 2$: polynomial

$\therefore$ any exponential function $y = b^x,\ b > 1$ grows at ∞ faster than any polynomial $y = x^p,\ p > 0$



$y = x^p,\ p > 1$

$y = mx$

$y = x^p,\ 0 < p < 1$

Powers of x vs. powers of ln x

Ex $f(x) = \ln x, \quad g(x) = x^p, \quad p > 0$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\ln x}{x^p} = \frac{\infty}{\infty} \underset{\text{Rule}}{\overset{\text{l'Hop.}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{p x^{p-1}} =$$

$$= \frac{1}{p} \lim_{x\to\infty} \frac{1}{x \cdot x^{p-1}} = \frac{1}{p} \lim_{x\to\infty} \frac{1}{x^{1+p-1}} =$$

$$x^a \cdot x^b = x^{a+b}$$

$$= \frac{1}{p} \lim_{x\to\infty} \frac{1}{x^p} = 0 \qquad \frac{x^a}{x^b} = x^{a-b}$$

$$\therefore \lim_{\substack{x\to\infty \\ p>0}} \frac{\ln x}{x^p} = 0 \Rightarrow$$ ln x grows at $x\to\infty$ slower than any polynomial

Ex $f(x) = \ln^q x, \quad g(x) = x^p, \quad p>0, q>0.$

$$\lim_{x\to\infty} \frac{\ln^q x}{x^p} = \lim_{x\to\infty} \left(\frac{\ln x}{x^{\frac{p}{q}}}\right)^q =$$

$$(x^a)^b = x^{a \cdot b} \qquad p = \frac{p}{q} \cdot q$$

$$\frac{a^q}{b^q} = \left(\frac{a}{b}\right)^q$$

$$= \left(\lim_{\substack{x\to\infty \\ \frac{p}{q}>0}} \frac{\ln x}{x^{p/q}}\right)^q = 0^q = 0$$

by previous example

$$\lim_{x\to\infty} \frac{\ln^q x}{x^p} = 0, \quad q, p > 0$$

$\therefore$ polynomials have greater growth rate than any power of logarithmic function

<u>Ex</u> <u>Powers of x vs. exponential function e^x</u>

$f(x) = x^p, \quad g(x) = e^x, \quad p > 0$

$$\lim_{x\to\infty} \frac{x^p}{e^x} = \left| \begin{array}{l} x = \ln t \\ e^x = e^{\ln t} = t \\ x\to\infty \Rightarrow t\to\infty \end{array} \right| = \lim_{t\to\infty} \frac{\ln^p t}{t} = 0$$

by previous example

$\therefore$ exp function grows faster than any polynomial function

<u>Ranking Growth Rates as $x\to\infty$</u>

<u>Notation:</u> $f \ll g$, we say that g grows faster than f as $x\to\infty$

with $p, q, r, s > 0$ and $b > 1$

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

$$x^p \cdot x^s$$

8.1 Integration by parts

Recall product rule:

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$$

$\int$ both sides.

$$u(x)v(x) = \int \left[u'(x)v(x) + u(x)v'(x)\right] dx$$

$$\Rightarrow \int u(x)\underbrace{v'(x)\,dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)\,dx}_{du}$$

Compact form: $du = u'(x)dx$

$dv = v'(x)\,dx$

$$\Rightarrow \boxed{\int u\,dv = uv - \int v\,du}$$

Integration by parts

Ex $\int \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv} = \left| \begin{matrix} u = x & dv = e^x dx \\ du = 1\cdot dx & v = e^x \end{matrix} \right| =$

$$= \underset{u}{x}\,\underset{v}{e^x} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du} = xe^x - e^x + C$$

Wrong choice : $\int x e^x dx = \int \underbrace{e^x}_{u} \underbrace{x\,dx}_{dv}$

$$\left| \begin{array}{ll} u = e^x & dv = x\,dx \\ du = e^x dx & v = \frac{x^2}{2} \end{array} \right| = \underbrace{e^x}_{u} \cdot \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{e^x dx}_{du} \Downarrow$$

more complicated $\int$

Ex $\int \underbrace{x}_{u} \underbrace{\cos x\, dx}_{dv} = \left| \begin{array}{ll} u = x & dv = \cos x\, dx \\ du = dx & v = \sin x \end{array} \right| =$

$$= \underbrace{x}_{u} \cdot \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \cdot \underbrace{dx}_{du} = x \sin x + \cos x + C$$

Ex $I = \int e^x \underbrace{\sin x}_{u}\, dx = \left| \begin{array}{ll} u = \sin x & dv = e^x dx \\ du = \cos x\, dx & v = e^x \end{array} \right| =$

$$= \sin x \cdot e^x - \int \underbrace{e^x}_{v} \underbrace{\cos x\, dx}_{du} = \left| \begin{array}{ll} u = \cos x & dv = e^x dx \\ du = -\sin x\, dx & v = e^x \end{array} \right|$$

$$= \sin x \cdot e^x - \left[e^x \cos x \underbrace{+}_{-(-)} \underbrace{\int e^x \sin x\, dx}_{I} \right]$$

Then

$$I = \sin x \cdot e^x - e^x \cos x - I \quad : \quad \text{equation for } I$$

Solve for I:

$$2I = \sin x \cdot e^x - e^x \cos x$$

$$\boxed{I = \frac{1}{2}\left[\sin x \cdot e^x - e^x \cos x\right] + C}$$

Ex

$$\int \underbrace{\ln x}_{u}\, dx = \left| \begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{dx}{x} & v = x \end{array} \right| =$$

$$= \ln x \cdot x - \int x \cdot \frac{dx}{x} = x \ln x - \int dx =$$

$$= x \ln x - x + C$$