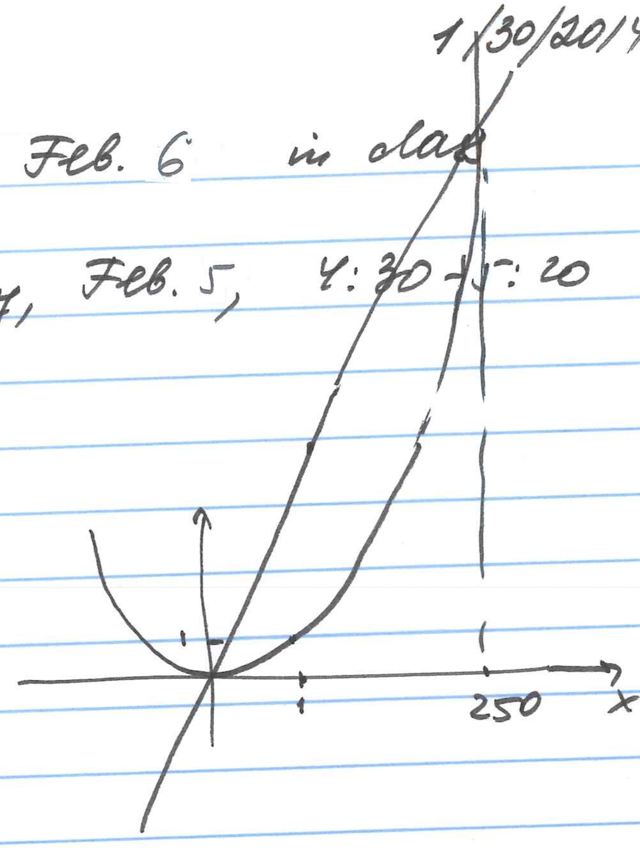
1/30/2014

Exam #1: Thursday, Feb. 6 in class

Review session: Wednesday, Feb. 5, 4:30–5:20

$y_1 = x^2$ $y_2 = 250x$



$x \neq 0$ $\frac{1}{x}$ $x^2 > 250x$

$x > 0$ $x > 250$

$y_1 = 2^x$, $y_2 = x^4$

$2^x > x^4$

in Matlab

```
>>   x = -1 : 0.01 : 300;
                  step
     y1 = x.^2;
     y2 = 250*x;
     plot (x,y1)
     hold on
     plot(x,y2,'r')
```

8.1 integration by parts: Definite integrals

Recall

$$\int u(x)\,v'(x)\,dx = u(x)v(x) - \int u'(x)\,v(x)\,dx$$

For definite integrals:

$$\int_a^b u(x)\,v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)\,v(x)\,dx$$

Ex

$$\int_0^{\ln 2} \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = \left| \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix} \right| =$$

$$= (xe^x)\Big|_{x=0}^{x=\ln 2} - \int_0^{\ln 2} \underbrace{e^x}_{v}\,dx =$$

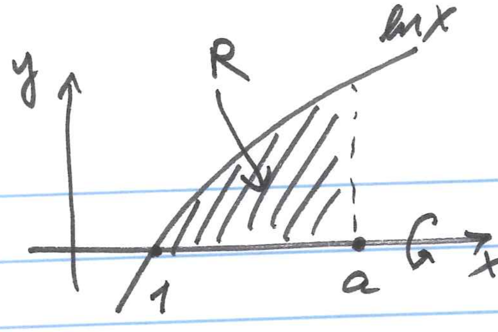
$$= \ln 2\,\underbrace{e^{\ln 2}}_{=2} - 0\cdot e^0 - e^x\Big|_{x=0}^{x=\ln 2} =$$

$$= 2\ln 2 - (\underbrace{e^{\ln 2}}_{2} - \underbrace{e^0}_{1}) = 2\ln 2 - 2 + 1 =$$

$$= 2\ln 2 - 1$$

Ex Solids of revolution

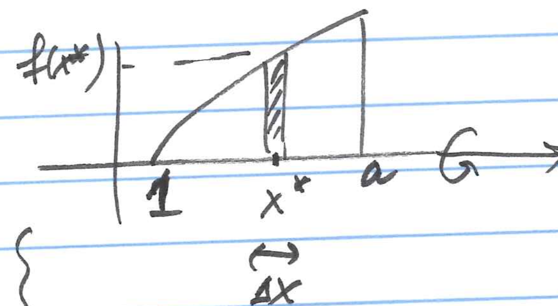
Let R be the region bounded by $y = \ln x$, x-axis and the line $x = a$, $a > 1$.



Find the volume of the solid that is obtained by revolving region R about x-axis.

We will use the method of disk (disk method)

$$V = \pi \int_1^a (\ln x)^2 \, dx \;=$$



$$V = \lim_{k \to \infty} \sum_{k=1}^{n} \pi \left(f(x_k^*)\right)^2 \Delta x$$

Use integration by parts.

$$\begin{vmatrix} u = (\ln x)^2 & dv = dx \\ du = 2\ln x \cdot \frac{1}{x}\, dx & v = x \end{vmatrix}$$

$$= \pi \left[x(\ln x)^2 \Big|_1^a - \int_1^a \cancel{x} \cdot 2\ln x \, \frac{1}{\cancel{x}} \, dx \right] =$$

$$= \pi \left[x(\ln x)^2 \Big|_1^a - 2\int_1^a \ln x \, dx \right] \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$= \begin{vmatrix} u = \ln x & dv = dx \\ du = \frac{dx}{x} & v = x \end{vmatrix} = \pi \left[x(\ln x)^2 \Big|_1^a - \right.$$

$$\left. - 2\left\{ x \ln x \Big|_1^a - \int_1^a \underbrace{x}_{v} \underbrace{\frac{dx}{x}}_{du} \right\} \right. =$$

$$= \pi\left[x(\ln x)^2 \Big|_1^a - 2\left\{ x \ln x \Big|_1^a - x \Big|_1^a \right\} \right] =$$

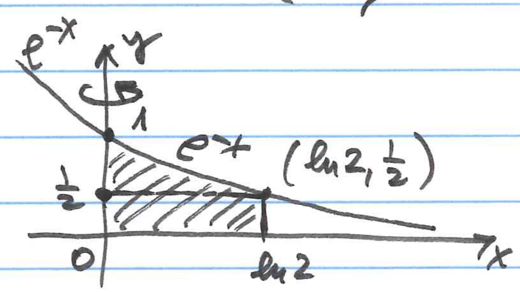
(*) $$= \pi\left[x(\ln x)^2 \Big|_1^a - 2\left\{ x \ln x - x \right\} \Big|_1^a \right] =$$

$$= \pi\left[a(\ln a)^2 - 2\left\{ a \ln a - (a-1) \right\} \right]$$

since $\ln 1 = 0$

$$x(\ln x)^2 \Big|_1^a = a(\ln a)^2 - 1(\cancel{\ln 1})^2$$ $\to 0$

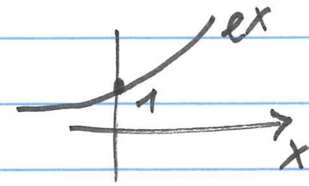
<u>Ex</u> #37 The region R is bounded by $f(x) = e^{-x}$, $x = \ln 2$ and the y-coordinate axis, and revolved about y-axis. Find volume of solid of revolution.



$y = e^{-x} \Rightarrow x = -\ln y$

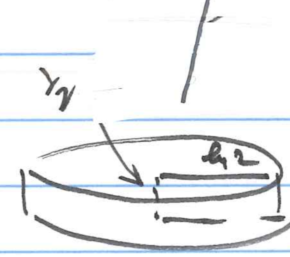
$x = \ln 2 \Rightarrow y = e^{-\ln 2} = e^{\ln 2^{-1}}$
$= 2^{-1} = \frac{1}{2}$

$n \ln x = \ln x^n$



$V = V_1 + V_2$

$V_1 = \pi \int_{\frac{1}{2}}^{1} (-\ln y)^2\, dy$, $\quad V_2 = \pi \cdot (\ln 2)^2 \cdot \frac{1}{2}$



$$V_1 = \pi \int_{\frac{1}{2}}^{1} (\ln y)^2 \, dy \overset{\text{use}}{=}$$

previous example
$1 \to \frac{1}{2}$
$a \to 1$
eq^n $(*)$

$$= \pi \left[y(\ln y)^2 \Big|_{\frac{1}{2}}^{1} - 2 \{ y \ln y - y \} \Big|_{\frac{1}{2}}^{1} \right] =$$

$\ln 1 = 0$

$$= \pi \left[-\frac{1}{2}\left(\ln \frac{1}{2}\right)^2 - 2 \left\{ -\frac{1}{2} \ln \frac{1}{2} - \left(1 - \frac{1}{2}\right) \right\} \right] =$$

$$\boxed{\ln \frac{1}{2}} = \cancelto{0}{\ln 1} - \ln 2 = \boxed{-\ln 2}$$

$\ln \frac{a}{b} = \ln a - \ln b$

$\ln(ab) = \ln a + \ln b$

$$= \pi \left[-\frac{1}{2} (-\ln 2)^2 - 2 \left\{ \frac{1}{2} \ln 2 - \frac{1}{2} \right\} \right] =$$

$\ln \frac{1}{2} = -\ln 2$

$$= \pi \left[-\frac{1}{2} (\ln 2)^2 - (\ln 2 - 1) \right] = V_1$$

$$V = V_1 + V_2 = \pi \left[\cancel{-\frac{1}{2}(\ln 2)^2} - (\ln 2 - 1) \right] + \cancel{\pi \frac{1}{2} (\ln 2)^2} =$$

$$= \boxed{\pi (1 - \ln 2)}$$

$\underline{\underline{Ex}}$ $\int x \sin x \cos x \, dx = \frac{1}{2}\int x \sin 2x \, dx \overset{\text{by}}{\underset{\text{parts}}{=}} \left| \begin{array}{ll} u = x & dv = \sin 2x \, dx \\ du = dx & v = -\frac{1}{2}\cos 2x \end{array} \right|$

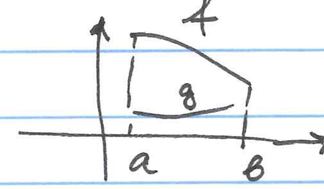
$\sin 2x = 2 \sin x \cos x$

$= -\frac{1}{2}\cos 2x \cdot x + \frac{1}{2}\int \cos 2x \, dx = -\frac{1}{2} x \cos 2x +$

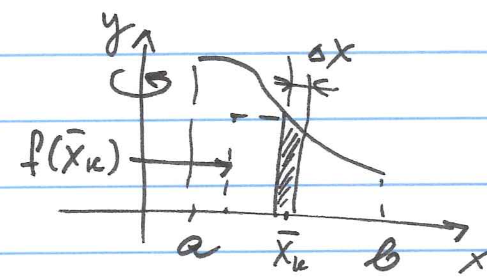
$+ \frac{1}{4}\sin 2x + C$

Volume by the Shell Method

$$V = \int_a^b 2\pi x \left(f(x) - g(x)\right) dx$$

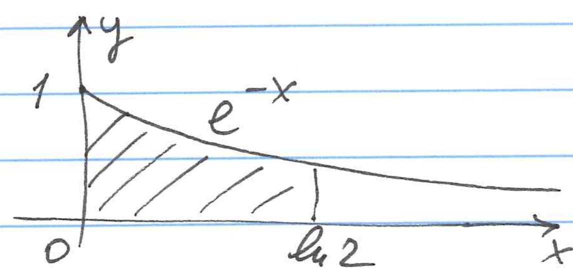


$$V = \int_a^b 2\pi x \, f(x) \, dx$$



$$\underbrace{2\pi \bar{x}_k}_{\text{length}} \cdot \underbrace{f(\bar{x}_k)}_{\text{height}} \underbrace{\Delta x}_{\text{thickness}}$$

#37



$$V = \int_0^{\ln 2} 2\pi x \cdot e^{-x} dx = 2\pi \int_0^{\ln 2} x e^{-x} dx \overset{\text{by}}{\underset{\text{parts}}{=}}$$

$$\left| \begin{array}{ll} u = x & dv = e^{-x} dx \\ du = dx & v = -e^{-x} \end{array} \right| = 2\pi \left[-x e^{-x} \Big|_0^{\ln 2} + \right.$$

$$\left. + \int_0^{\ln 2} e^{-x} dx \right] = 2\pi \left[-\ln 2 \, e^{-\ln 2} + 0 - e^{-x} \Big|_0^{\ln 2} \right]$$

$$= 2\pi \left[-\ln 2 \cdot \frac{1}{2} - (\underbrace{e^{-\ln 2}}_{\frac{1}{2}} - 1) \right] = 2\pi \left[-\frac{1}{2} \ln 2 + \frac{1}{2} \right]$$

$$= \pi \left[1 - \ln 2 \right]$$

8.2 Trigonometric Integrals

Integrating Powers of cos x and sin x

$\int \sin^m x\,dx$, $\int \cos^n x\,dx$, $m, n > 0$ integers

odd power $m=5$

Ex $\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx \ \textcircled{=}$

$(\cos x)^5$ — split off a single factor

$\cos^2 x = 1 - \sin^2 x$ since $\cos^2 x + \sin^2 x = 1$

$\cos^4 x = (1 - \sin^2 x)^2$

$\textcircled{=} \int (1 - \sin^2 x)^2 \cos x\,dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array} \right| =$

$= \int (1 - u^2)^2\,du = \int (1 - 2u^2 + u^4)\,du \ \textcircled{=}$

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$\textcircled{=}\ u - \frac{2}{3}u^3 + \frac{u^5}{5} + C = \sin x - \frac{2}{3}\sin^3 x +$

$+ \frac{1}{5}\sin^5 x + C$

Half-angle formulas

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}}$$

even power Ex

$$\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx =$$

use half-angle formulas

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) dx =$$

half-angle formula

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}\left(1+\cos 4x\right)\right) dx =$$

$$= \frac{1}{4}\left(\underline{x} - \sin 2x + \frac{1}{2}\left(\underline{x} + \frac{1}{4}\sin 4x\right)\right) + C =$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$x + \frac{1}{2}x = x\left(1+\frac{1}{2}\right) = \frac{3}{2}x$$

Integrating Products of $\sin x$ and $\cos x$

$$\int \sin^m x \cos^n x$$

odd power: split off one single factor

both even: use half-angle formulas

Ex $\int \sin^4 x \cos^2 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$

4, 2: both even powers ⇒ use half-angle formulas

$$= \frac{1}{8}\int (1 - 2\cos 2x + \cos^2 2x)(1+\cos 2x)\, dx =$$

$$= \frac{1}{8}\int \left(1 + \cos 2x - 2\cos 2x - 2\cos^2 2x + \cos^2 2x + \right.$$

$$\left. + \cos^3 2x\right) dx = \frac{1}{8}\int \left(1 - \cos 2x - \cos^2 2x + \right.$$

half-angle

$$\left. + \cos^3 2x\right) dx = \frac{1}{8}\int (1-\cos 2x)\, dx -$$

split off a factor

$$- \frac{1}{8}\int \underbrace{\frac{1+\cos 4x}{2}}_{\cos^2 2x} dx + \frac{1}{8}\int \overbrace{\underbrace{\cos^2 2x \cdot \cos 2x}_{\cos^3 2x}\, dx}^{I} =$$

$$I = \int \cos^2 2x \cdot \cos 2x \, dx = \int (1 - \sin^2 2x)\underbrace{\cos 2x\, dx}_{\frac{1}{2}du} =$$

$$= \left| \begin{array}{l} u = \sin 2x \\ du = 2\cos 2x\, dx \end{array} \right| = \frac{1}{2}\int (1-u^2)\, du =$$

$$= \frac{1}{2}\left(u - \frac{u^3}{3}\right) + C_1 = \frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + C_1$$

$$= \frac{1}{8}\left(x - \cancel{\frac{1}{2}\sin 2x}\right) - \frac{1}{16}\left(x + \frac{1}{4}\sin 4x\right) +$$

$$+ \frac{1}{8}\left[\frac{1}{2}\left(\cancel{\sin 2x} - \frac{\sin^3 2x}{3}\right)\right] + C =$$

$$\frac{1}{8}x - \frac{1}{8}\frac{1}{2}x = \frac{1}{8}x\left(1 - \frac{1}{2}\right) = \frac{1}{8}\frac{1}{2}x$$

$$= \frac{1}{8 \cdot 2}x - \frac{1}{8 \cdot 2 \cdot 4}\sin 4x - \frac{1}{8 \cdot 2 \cdot 3}\sin^3 2x + C =$$

$$= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right) + C$$

Ex $\int \sin^3 x \cos^{-2} x \, dx = \int \sin^2 x \cdot \sin x \, \frac{1}{\cos^2 x} \, dx =$

split off

$$= \int (1 - \cos^2 x) \frac{1}{\cos^2 x} \sin x \, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right| =$$

$$= -\int \frac{1 - u^2}{u^2} \, du = \int \frac{u^2 - 1}{u^2} \, du = \int \left(1 - \frac{1}{u^2}\right) du =$$

$$= u - \frac{1}{-2+1} u^{-2+1} + C = u + \frac{1}{u} + C =$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$= \cos x + \frac{1}{\cos x} + C =$$

$$= \cos x + \sec x + C$$

Reduction formulas

$n > 0$, integer

1. $$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

2. $$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

3. $$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \qquad n \neq 1$$

4. $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \qquad n \neq 1$$

Ex $$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \left| \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right| =$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C =$$

$$n \ln x = \ln x^n$$

$$= \ln |(\cos x)^{-1}| + C = \ln|\sec x| + C$$

$$\boxed{\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C}$$

Ex $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array} \right| =$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

$$\boxed{\int \cot x\,dx = \ln|\sin x| + C}$$

Ex $\displaystyle\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx =$

$$= \int \frac{\cos x}{1-\sin^2 x}\,dx = \left| \begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array} \right| = \int \frac{du}{1-u^2}$$

Partial fraction decomposition

$$\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{\overset{(1+u}{A}}{1-u} + \frac{\overset{(1-u}{B}}{1+u} =$$

$$= \frac{A(1+u) + B(1-u)}{(1-u)(1+u)}$$

$$\therefore \quad 1 = A(1+u) + B(1-u)$$

2/3/2014

$$\frac{1}{1-u^2} = \frac{A(1+u) + B(1-u)}{(1-u)(1+u)}$$

$$\therefore \quad 1 = A(1+u) + B(1-u)$$

$$= \quad 1 = (A-B)u + (A+B)$$

$0 \cdot u + 1$

We have two polynomials: on the left and on the right. They are equal $\Rightarrow$ their coefficients of like powers should be the same.

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

n^{th} order polynomial

$a_n \neq 0$

$$\left.\begin{array}{ll} u = u^1: & 0 = A - B \\ 1 = u^0: & 1 = A + B \end{array}\right\} \begin{array}{l} \Rightarrow \quad B = A \\ \Rightarrow \quad 1 = A + A \end{array}$$

constant term

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

$$B = A = \frac{1}{2}$$

$$\therefore \quad \frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u} = \frac{\frac{1}{2}}{1-u} + \frac{\frac{1}{2}}{1+u}$$

$$= \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)$$

$$\int \sec x\,dx = \int \frac{du}{1-u^2} \;=\;$$

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)$$: partial fraction decomposition

$$= \frac{1}{2}\left[\int \frac{du}{1-u} + \int \frac{du}{1+u}\right] \;\equiv$$

$$\int \frac{du}{1-u} = \left| \begin{array}{l} t = 1-u \\ dt = -du \end{array} \right| = \int \frac{-dt}{t} = -\ln|t| + C$$

$$= -\ln|1-u| + C$$

$$\equiv \frac{1}{2}\left[-\ln|1-u| + \ln|1+u|\right] + C \;=$$

$$\ln a - \ln b = \ln\frac{a}{b}$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right| + C \;=$$

$$(a-b)(a+b) = a^2 - b^2$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C =$$

$$= \frac{1}{2} \ln \left| \frac{(1+\sin x)^2}{\cos^2 x} \right| + C =$$

$a \ln x = \ln x^a$

$$= \ln \left| \frac{(1+\sin x)^2}{\cos^2 x} \right|^{1/2} + C =$$

$$= \ln \left| \frac{1+\sin x}{\cos x} \right| + C = \ln |\sec x + \tan x| + C$$

$$\therefore \boxed{\int \sec x = \ln|\sec x + \tan x| + C}$$

Similarly,

$$\boxed{\int \csc x \, dx = -\ln|\csc x + \cot x| + C}$$

<u>Integrating Products of tan x and sec x</u>

$$\int \tan^m x \sec^n x \, dx$$

Ex Evaluate $\int \tan^4 x \, dx$

4: even power

Recall $\cos^2 x + \sin^2 x = 1 \quad \Big| \, \frac{1}{\cos^2 x}$

$$\boxed{1 + \tan^2 x = \sec^2 x}$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \cdot \tan x$$

$$\int \tan^4 x \, dx = \int \tan^2 x \, (\underbrace{\sec^2 x - 1}_{\tan^2 x}) \, dx =$$

$$= \underbrace{\int \tan^2 x \cdot \sec^2 x \, dx}_{\text{I}} - \underbrace{\int \tan^2 x \, dx}_{\text{II}} \;\boxed{=}$$

$$\text{I} = \int \tan^2 x \cdot \underbrace{\sec^2 x \, dx}_{du} = \left| \begin{matrix} u = \tan x \\ du = \sec^2 x \, dx \end{matrix} \right| =$$

$$= \int u^2 \, du = \frac{u^3}{3} + C_1 = \frac{1}{3} \tan^3 x + C_1$$

$$\text{II} = \int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C_2$$

$$\boxed{=} \; \frac{1}{3} \tan^3 x - \tan x + x + C$$

alternatively, we can reduction formula

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx \;;\quad n \neq 1$$

$$\Rightarrow \underbrace{\int \tan^4 x\,dx}_{n=4} = \frac{\tan^3 x}{3} - \underbrace{\int \tan^2 x\,dx}_{n=2} =$$

$$= \frac{1}{3}\tan^3 x - \left[\frac{\tan x}{2-1} - \int \underset{\substack{\uparrow \\ \tan^0 x}}{1}\,dx\right] =$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Ex $\int \tan^3 x \sec^4 x\,dx =$

4: even power of sec
split off $\sec^2 x$

use $u = \tan x$ since $du = \sec^2 x\,dx$

$$= \int \tan^3 x \underbrace{\sec^2 x}_{1+\tan^2 x} \sec^2 x\,dx =$$

$$= \int \tan^3 x\,(1+\tan^2 x)\,\underbrace{\sec^2 x\,dx}_{du} =$$

$$= \left| \begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array} \right| = \int u^3 (1+u^2)\, du =$$

$$= \int (u^3 + u^5)\, du = \frac{u^4}{4} + \frac{u^6}{6} + C =$$

$$= \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C$$

Ex $\int \underbrace{\tan^2 x}_{\sec^2 x - 1} \sec x\, dx = \int (\sec^2 x - 1) \sec x\, dx =$

$$= \underbrace{\int \sec^3 x\, dx}_{I} - \underbrace{\int \sec x\, dx}_{II} =$$

For I we can use reduction formula

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx \qquad n \neq 1$$

$$I = \int \underbrace{\sec^3 x\, dx}_{n=3} = \frac{\sec x \cdot \tan x}{2} + \frac{1}{2} \int \sec x\, dx$$

$$= \frac{1}{2} \sec x \cdot \tan x + \underbrace{\frac{1}{2} \int \sec x\, dx - \int \sec x\, dx}_{= -\frac{1}{2} \int \sec x\, dx} =$$

$$= \frac{1}{2} \sec x \cdot \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C$$

Summary : $\int \tan^m x \sec^n x \, dx$

n even : split off $\sec^2 x$, rewrite the remaining even power of $\sec x$ in terms of $\tan x$
use $u = \tan x$ (since $du = \sec^2 x \, dx$)

m odd : split off $\sec x \tan x$, rewrite the remaining even power of $\tan x$ in terms of $\sec x$.
$u = \sec x$ (since $du = \sec x \tan x \, dx$)

m even, n odd : rewrite even powers of $\tan x$ in terms of $\sec x$. $\Rightarrow$ gives polynomial in $\sec x$. $\Rightarrow$ use reduction formula for $\int \sec^n x$.

2/4/2014

(#26) $\int \sec^{-2}x \tan^3 x\, dx = \int \sec^{-2}x \tan^2 x \cdot \tan x\, dx \;(=)$

Recall

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x \leftarrow$$

$$(=) \int \underbrace{\sec^{-2}x \cdot \frac{1}{\sec x}}_{=\frac{1}{\sec^3 x}} \overbrace{\tan^2 x}^{=\sec^2 x - 1} \cdot \underbrace{\sec x \cdot \tan x\, dx}_{du}$$

$1+\tan^2 x = \sec^2 x$

$$= \left| \begin{array}{l} u = \sec x \\ du = \sec x \tan x\, dx \end{array} \right| = \int \frac{u^2-1}{u^3}\, du =$$

$$= \int \left(\frac{1}{u} - \frac{1}{u^3}\right) du = \ln|u| - \frac{1}{-3+1}\frac{1}{u^2} + C =$$

$$= \ln|\sec x| + \frac{1}{2}\frac{1}{\sec^2 x} + C =$$

$$= \ln|\sec x| + \frac{1}{2}\cos^2 x + C$$

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C$$

Analogous techiques are available for

$$\int \cot^m x \csc^n x\, dx$$

Recall

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$$

#27 Evaluate $\int \frac{\csc^4 x}{\cot^2 x}\,dx = \int \csc^4 x \cdot \cot^{-2} x\,dx =$

4: even $\Rightarrow$ split off $\csc^2 x$

$$= \int \overbrace{\csc^2 x}^{\cot^2 x+1} \cdot \cot^{-2} x \cdot \overbrace{\csc^2 x\,dx}^{-du} \quad \ominus$$

$u = \cot x$

$du = -\csc^2 x\,dx$

$\cos^2 x + \sin^2 x = 1 \quad \Big| \frac{1}{\sin^2 x}$

$\cot^2 x + 1 = \csc^2 x$

$$\ominus -\int \frac{u^2+1}{u^2}\,du = -\int \left(1 + \frac{1}{u^2}\right) du =$$

$$= -\left(u - \frac{1}{u}\right) + C = -\left(\cot x - \frac{1}{\cot x}\right) + C =$$

$$= -\cot x + \tan x + C$$

8.3 Trigonometric Substitutions

Integrals involving a^2-x^2, a^2+x^2, x^2-a^2

(I) Integrals involving a^2-x^2, $a>0$

Substitution $x = a\sin\theta$: sine substitution

$$a^2-x^2 = a^2-(a\sin\theta)^2 = a^2-a^2\sin^2\theta = a^2(1-\sin^2\theta)$$
$$= a^2\cos^2\theta$$

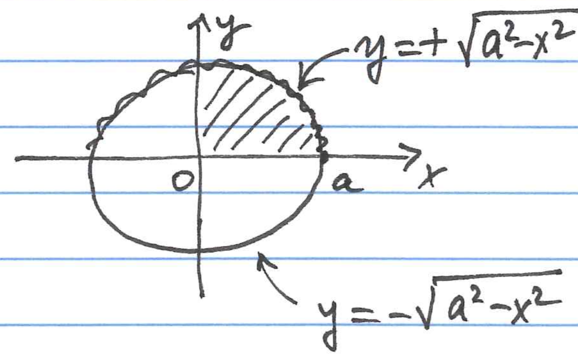
$$\Rightarrow \quad a^2-x^2 = a^2\cos^2\theta$$

__Ex__ Area of a circle of radius a : πa^2

$x^2+y^2=a^2$

$y^2=a^2-x^2$

$y=\pm\sqrt{a^2-x^2}$



$$\text{Area} = \int_0^a \sqrt{a^2-x^2}\,dx$$

$$\therefore \text{Area} = 4\int_0^a \sqrt{a^2-x^2}\,dx =$$

sine substitution
$x = a\sin\theta$
$dx = a\cos\theta\,d\theta$
$a^2-x^2 = a^2-a^2\sin^2\theta$
$= a^2\cos^2\theta$

$$x=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$$

$$\underset{\overset{\parallel}{a}}{x}=a\sin\theta \Rightarrow \cancel{a=a\sin\theta} \Rightarrow \sin\theta=1 \qquad x=a\sin\theta,\quad -\frac{\pi}{2}\le\theta<\frac{\pi}{2}$$

$$x=a \Rightarrow \sin\theta=1 \Rightarrow \theta=\frac{\pi}{2}$$

$$=4\int_0^{\pi/2}\sqrt{a^2\cos^2\theta}\cdot a\cos\theta\,d\theta = 4a^2\int_0^{\pi/2}|\cos\theta|\cos\theta\,d\theta \ \ominus$$

$$0\le\theta\le\frac{\pi}{2} \Rightarrow \cos\theta\ge 0 \Rightarrow |\cos\theta|=\cos\theta$$

$$\ominus\ 4a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = 4a^2\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=2a^2\left(\theta+\frac{1}{2}\cancelto{0}{\sin 2\theta}\right)\Big|_0^{\pi/2} = 2a^2\cdot\frac{\pi}{2}=\pi a^2 \checkmark$$

Ex

$$\int\frac{dx}{(16-x^2)^{3/2}} = \left|\begin{array}{l} \text{sine substitution} \\ x=4\sin\theta \\ dx=4\cos\theta\,d\theta \\ 16-x^2=16-16\sin^2\theta=16\cos^2\theta \end{array}\right|$$

$$a=4$$

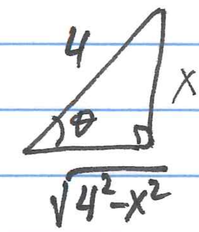
$$=\int\frac{4\cos\theta\,d\theta}{(16\cos^2\theta)^{\frac{3}{2}}} = \int\frac{4\cos\theta\,d\theta}{16^{\frac{3}{2}}(\cos^2\theta)^{\frac{3}{2}}} \ \ominus$$

$$\frac{4}{16^{\frac{3}{2}}}=\frac{2^2}{(2^4)^{\frac{3}{2}}}=\frac{2^2}{2^{4\cdot\frac{3}{2}}}=\frac{2^2}{2^6}=2^{2-6}=2^{-4}=\frac{1}{16}$$

$$\ominus\ \frac{1}{16}\int\frac{\cos\theta\,d\theta}{\cos^3\theta}=\frac{1}{16}\int\frac{d\theta}{\cos^2\theta}=\frac{1}{16}\int\sec^2\theta\,d\theta$$

$$= \frac{1}{16}\tan\theta + C \;=$$

$$x = 4\sin\theta \Rightarrow \sin\theta = \frac{x}{4}$$



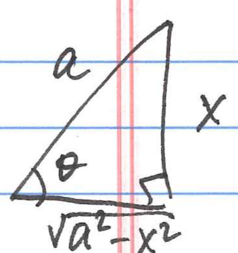
$$\tan\theta = \frac{x}{\sqrt{16-x^2}}$$

$$= \frac{1}{16}\frac{x}{\sqrt{16-x^2}} + C$$

Trigonometric substitutions

a^2-x^2 we use sine substitution

$$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$



$$a^2-x^2 = a^2-a^2\sin^2\theta = a^2\cos^2\theta$$

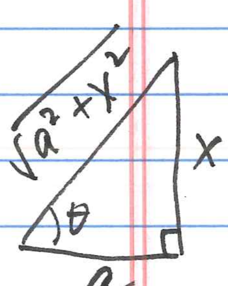
$$\sin\theta = \frac{x}{a}$$

---

a^2+x^2

$$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

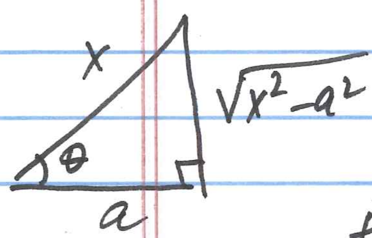
$$a^2+x^2 = a^2+a^2\tan^2\theta = a^2\sec^2\theta$$



$$\tan\theta = \frac{x}{a}$$

$x^2 - a^2$ $\qquad$ $x = a\sec\theta$ $\qquad$ $0 \le \theta < \frac{\pi}{2}$, $x \geq a$

$\frac{\pi}{2} < \theta \le \pi$, $x \le -a$



$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

$\sec\theta = \frac{x}{a}$

$\parallel$

$\frac{1}{\cos\theta} \Rightarrow \cos\theta = \frac{a}{x}$

2/7/2014

Ex Arc length of a parabola

$y = f(x), \quad a \le x \le b$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}$$

Find the arc length of the segment of parabola $y = x^2$ on $[0, 2]$.

$$L = \int_0^2 \sqrt{1 + 4x^2}\, dx = \qquad f(x) = x^2, \quad f'(x) = 2x$$

$\sqrt{a^2 + x^2}$

$$= \int_0^2 \sqrt{4\left(\frac{1}{4} + x^2\right)}\, dx = 2\int_0^2 \sqrt{\frac{1}{4} + x^2}\, dx \ (=)$$

$a = \frac{1}{2}$

= tangent substitution

$x = a \tan\theta$

$x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \tan\theta = 4$

$\theta = \arctan 4$

$$(=)\ 2\int_0^{\arctan 4} \sqrt{\frac{1}{4} + \frac{1}{4}\tan^2\theta}\ \frac{1}{2}\sec^2\theta\, d\theta =$$

$$= \cancel{2}\int_0^{\arctan 4} \cancel{\tfrac{1}{2}} \sqrt{\sec^2\theta} \cdot \tfrac{1}{2}\sec^2\theta \, d\theta = \tfrac{1}{2}\int_0^{\arctan 4} \sec^3\theta \, d\theta \;=$$

$$= \frac{1}{2}\int_0^{\arctan 4} \underbrace{\sec\theta}_{\sqrt{\sec^2\theta}} \sec^2\theta \, d\theta \qquad \cancel{\sec^3\theta = \frac{\sec^2\theta \cdot \sec\theta \tan}{\tan\theta}}$$

$$\cancel{\sqrt{\sec^2\theta} = \sqrt{1+\tan^2\theta}}$$

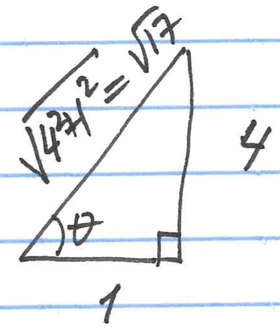
use reduction formula

$$= \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\arctan 4} =$$

$\arctan\theta = \tan^{-1}\theta$

$\theta = \arctan 4 \Rightarrow$

$\tan\theta = 4$



$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{\sqrt{17}}} = \sqrt{17}$$

$$= \frac{1}{4}\left(\sqrt{17}\cdot 4 + \ln|\sqrt{17}+4|\right) \approx 4.65$$

Tangent substitution

Ex Evaluate

$$\int \frac{dx}{(1+x^2)^2} = \left| \begin{array}{ll} x = \tan\theta & 1+x^2 = 1+\tan^2\theta \\ dx = \sec^2\theta \, d\theta & \quad = \sec^2\theta \end{array} \right|$$

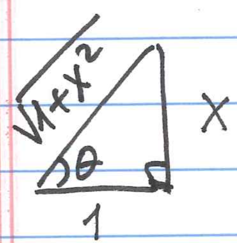
$a=1$

$$= \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2} = \int \frac{d\theta}{\sec^2\theta} = \int \cos^2\theta \, d\theta =$$

$$= \int \frac{1+\cos 2\theta}{2} \, d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \ (=)$$

$x = \tan\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$



$$\cos\theta = \frac{1}{\sqrt{1+x^2}}; \quad \sin\theta = \frac{x}{\sqrt{1+x^2}}$$

$$(=)\ \frac{1}{2}\left(\theta + \frac{1}{\cancel{2}} \cdot \cancel{2}\sin\theta\cos\theta\right) + C =$$

$$= \frac{1}{2}(\theta + \sin\theta\cos\theta) + C = \frac{1}{2}\left(\tan^{-1}x + \right.$$

$$\left. + \frac{x}{\sqrt{1+x^2}} \frac{1}{\sqrt{1+x^2}}\right) + C = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{1+x^2}\right) + C$$

A Secant Substitution

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2} \, dx \ (=)$$

$x^2 + 4x - 5 =$ $\qquad (a+b)^2 = a^2 + 2ab + b^2$

$$= \underbrace{x^2 + 2\cdot 2x + 4}\, - 4 - 5 = (x+2)^2 - 9$$

$$= \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2} = \left| \begin{array}{l} u = x+2 \\ du = dx \\ x=1 \Rightarrow u=3 \\ x=4 \Rightarrow u=6 \end{array} \right| =$$

$$= \int_3^6 \frac{\sqrt{u^2-3^2}}{u}\, du = \left| \begin{array}{l} u = a\sec\theta \\ u = 3\sec\theta \\ du = 3\sec\theta\tan\theta\, d\theta \\ u^2-3^2 = 3^2\sec^2\theta - 3^2 = \\ \quad = 3^2\tan^2\theta \end{array} \right|$$

$a=3$

$$u=3 \Rightarrow 3 = 3\sec\theta$$
$$\sec\theta = 1 \Rightarrow \cos\theta = 1$$
$$\boxed{\theta = 0}$$

$$u=6 \Rightarrow 6 = 3\sec\theta$$
$$\sec\theta = 2$$
$$\cos\theta = \frac{1}{2}$$
$$\boxed{\theta = \frac{\pi}{3}}$$

$$= \int_0^{\pi/3} \frac{\sqrt{3^2\tan^2\theta}}{\cancel{3\sec\theta}} \cancel{3\sec\theta}\tan\theta\, d\theta$$

$$= 3\int_0^{\pi/3} \tan\theta \cdot \tan\theta\, d\theta$$

$$= 3\int_0^{\pi/3} \underbrace{\tan^2\theta}_{\sec^2\theta - 1}\, d\theta =$$

$$= 3\int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta = 3(\tan\theta - \theta)\Big|_0^{\pi/3}$$

$$= 3\left(\sqrt{3} - \frac{\pi}{3}\right)$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \frac{1}{2}$$

8.4 Partial Fractions

$(x+4 \qquad (x-2$

$$f(x) = \frac{1}{x-2} + \frac{2}{x+4} = \frac{x+4+2(x-2)}{(x-2)(x+4)} =$$

fractions

$$= \frac{3x}{x^2+2x-8} \quad : \text{rational function}$$

Recall, a rational function $f(x) = \frac{p(x)}{g(x)}$ is a ratio of two polynomials $p(x)$ and $g(x)$.

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + \overbrace{a_1 x + a_0}$$

polynomial of degree n

$a_n \neq 0$

Then we can write

$$\frac{3x}{x^2+2x-8} = \frac{1}{x-2} + \frac{2}{x+4} \quad : \text{partial fraction decomposition}$$

$$\int \frac{3x\,dx}{x^2+2x-8} = \int \left(\frac{1}{x-2} + \frac{2}{x+4} \right) dx$$

difficult to do   easy to compute

Ex

$$\frac{3x}{x^2+2x-8}$$

Vietta's Thm

$ax^2+bx+c=$

$=a(x-x_1)(x-x_2)$

x_1, x_2: roots

$x_1+x_2=-\frac{b}{a}$

$x_1 \cdot x_2 = \frac{c}{a}$

$x_1+x_2=-2$

$x_1 \cdot x_2 = -8$

$2, -4$

$x_1=2, \ x_2=-4$

$$x_{1,2}=\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{4+4 \cdot 8}}{2} =$$

$$= \frac{-2 \pm \sqrt{4 \cdot 9}}{2} = \frac{-2 \pm 6}{2} = -1 \pm 3 = \left[\begin{array}{l} +2 \\ -4 \end{array}\right.$$

$\Rightarrow x^2+2x-8 = (x-2)(x-(-4))$

$$\therefore \quad \frac{3x}{x^2+2x-8} = \frac{3x}{(x-2)(x+4)} =$$

$$= \frac{A^{(x+4}}{x-2} + \frac{B^{(x-2}}{x+4} \; = \quad \text{partial fraction decomposition}$$

A, B: undetermined coefficients

$$\frac{3x}{x^2+2x-8} = \frac{A(x+4)+B(x-2)}{(x-2)(x+4)}$$

$$\therefore \ 3x = A(x+4)+B(x-2)$$

$$3x = (A+B)x + (4A - 2B)$$

$1 = x^0$: $\quad 0 = 4A - 2B \Rightarrow 2B = 4A$

$$B = 2A$$

$x = x^1$: $\quad 3 = A + B$

$$\Rightarrow 3 = A + 2A$$

$$3 = 3A \Rightarrow \boxed{A = 1}$$

$$\boxed{B = 2A = 2}$$

$$\therefore \frac{3x}{x^2+2x-8} = \frac{1}{x-2} + \frac{2}{x+4}$$