2/10/2014

Partial Fraction Decomposition (Cont'd)

Simple Linear Factors

$f(x) = \dfrac{p(x)}{q(x)}$, $p(x), q(x)$: polynomials

$\deg p < \deg q$

p, q have no common factors

if $q(x) = (x - r_1)(x - r_2) \dots (x - r_n)$

$r_1, r_2, \dots, r_n$: real n roots

then

$$(*) \quad f(x) = \frac{p(x)}{q(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \dots + \frac{A_n}{x - r_n}$$

where $A_1, \dots, A_n$ are to be determined. Multiply both sides of $(*)$ by $q(x)$. This gives system of equations to find $A_1, \dots, A_n$. Alternatively, find the least common denominator on RHS of $(*)$. It is the same as $q(x)$. Then numerators $p(x)$ and numerator on RHS are equal. This gives equation with polynomials on both sides. Two polynomials are equal if coefficients of like powers are equal. This gives a system of equations for $A_1, \dots, A_n$.

Repeated Linear Factors

Ex

$$f(x) = \frac{5x^2-3x-2}{x^3-2x^2} = \frac{5x^2-3x-2}{x^2(x-2)} =$$

$$= \frac{A^{(x(x-2)}}{x} + \frac{B^{(x-2}}{x^2} + \frac{C^{(x^2}}{x-2} =$$

$$= \frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)}$$

$$\therefore \; 5x^2-3x-2 = Ax(x-2)+B(x-2)+Cx^2$$

We equate coefficients of like powers

$$5x^2-3x-2 = (A+C)x^2 + (-2A+B)x - 2B$$

x^2: $\; 5 = A+C \;\Rightarrow\; C = 5-A = 5-2 = \boxed{3 = C}$

x^1: $\; -3 = -2A+B \;\Rightarrow\; -3 = -2A+1$

$\qquad -4 = -2A \;\Rightarrow\; \boxed{A=2}$

x^0: $\; -2 = -2B \;\Rightarrow\; \boxed{B=1}$

Hence,

$$\frac{5x^2-3x-2}{x^2(x-2)} = \frac{2}{x} + \frac{1}{x^2} + \frac{3}{x-2}$$

$$f(x) = \frac{p(x)}{q(x)(x-r)^m} = \cdots + \frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}$$

r: repeated root of multiplicity m

$A_1, A_2, \ldots, A_m$: constants to be determined

Irreducible Quadratic Factors

ax^2+bx+c: has no real roots

$$f(x) = \frac{p(x)}{q(x)(ax^2+bx+c)} = \cdots + \frac{Ax+B}{ax^2+bx+c}$$

ax^2+bx+c: polynomial of degree 2

$Ax+B$: polynomial of degree 1 less, i.e. polynomial of degree 1

Ex $\dfrac{x^2+1}{x^4-4x^3-32x^2} = \dfrac{x^2+1}{x^2(x^2-4x-32)}$

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Ex $$\frac{x^2+1}{x^4-4x^3-32x^2} = \frac{x^2+1}{x^2(x^2-4x-32)} \ominus$$

$x^2-4x-32$

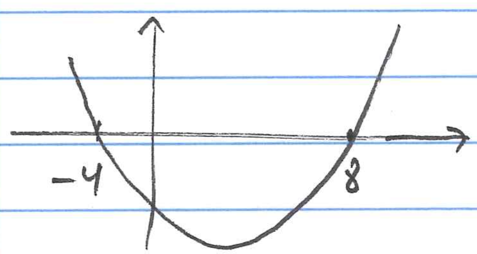
$8, -4$ $\qquad 8\cdot(-4)=-32$ ✓

$8+(-4)=4=-(-4)=-b$

$$x_{1,2}=\frac{4\pm\sqrt{4^2+4\cdot 32}}{2}=\frac{4\pm 4\sqrt{1+8}}{2}=2\pm 2\cdot 3=\begin{cases}8\\-4\end{cases}$$

$4\cdot 32=4^2\cdot 8$

$\therefore\ x^2-4x-32=(x-8)(x+4)$: reducible quadratic polynomial since it has real roots



$$\ominus\ \frac{x^2+1}{x^2(x-8)(x+4)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-8}+\frac{D}{x+4}$$

To find A, B, C, D, we find the least common denominator on RHS. This gives us a ratio of polynomials (rational function) and we equate numerators to get equality of two polynomials and equate like coefficients to get equations for A, B, C, D.

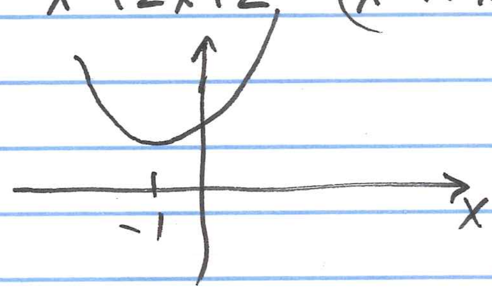
Aside: $\frac{2}{x}^{(x} + \frac{3x-1}{x^2} = \frac{2x+3x-1}{x^2} = \frac{5x-1}{x^2}$

$$\frac{1}{x} + \frac{3x+2}{x^2} + \frac{x-5}{x^3} = \frac{\cdots}{x^3}$$

Ex $\dfrac{10}{(x-2)^2(x^2+2x+2)} =$

$$x^2+2x+2 = (x^2+2x+1)+1 = (x+1)^2+1 > 0$$

has no real roots



Note: parabola is above x-axis and never crosses x-axis

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2-4\cdot 2}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$2^2 - 4\cdot 2 = 4-8 = -4 < 0$$

Hence, x^2+2x+2 is irreducible

$$= \frac{A}{x-2}^{(x-2)(x^2+2x+2)} + \frac{B}{(x-2)^2}^{(x^2+2x+2)} + \frac{Cx+D}{x^2+2x+2}^{((x-2)^2} \;\boxed{=}$$

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 :$$

n^{th} degree polynomial

$$P_2(x) = a_2x^2 + a_1x + a_0$$

$$P_1(x) = a_1x + a_0; \qquad P_0(x) = a_0$$

$$= \frac{A(x-2)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x-2)^2}{(x-2)^2(x^2+2x+2)}$$

$$\Rightarrow \quad 10 = A(x-2)(x^2+2x+2) + B(x^2+2x+2) + \\ + (Cx+D)(x-2)^2$$

$$10 = A(x^3 + \cancel{2x^2} + 2x - \cancel{2x^2} - 4x - 4) + B(x^2 + 2x + 2) \\ + (Cx+D)(x^2 - 4x + 4)$$

$$\left.\begin{aligned} x^3: &\quad 0 = A + C \\ x^2: &\quad 0 = B - 4C + D \\ x^1: &\quad 0 = -2A + 2B + 4C - 4D \\ x^0: &\quad 10 = -4A + 2B + 4D \end{aligned}\right\} \text{solve for } A, B, C, D$$

$\underline{Ex}$

$$\frac{3x+2}{(x-2)^4(x^2+2x+2)^3} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \\ + \frac{A_4}{(x-2)^4} + \frac{C_1x + D_1}{x^2+2x+2} + \frac{C_2x+D_2}{(x^2+2x+2)^2} + \frac{C_3x+D_3}{(x^2+2x+2)^3}$$

$$= \frac{1}{x-2} + \frac{2}{x+4}$$

Ex $$\frac{3x}{x^2+2x-8} = \frac{A}{x-2} + \frac{B}{x+4} \quad \Big| \cdot (x-2)$$

Multiply both sides by $(x-2)$ and take $\lim_{x \to 2}$

$$\frac{3x\cancel{(x-2)}}{\cancel{(x-2)}(x+4)} = A + \frac{B(x-2)}{x+4}$$

$$\lim_{x\to 2} \frac{3x}{x+4} = A + \cancel{\lim_{x\to 2} B\frac{(x-2)}{x+4}} \to 0$$

$$\frac{3\cdot 2}{2+4} = A \Rightarrow A = \frac{3\cdot 2}{6} = 1 \quad \checkmark$$

Multiply both sides by $(x+4)$ and take $\lim_{x\to -4}$

$$\frac{3x\cdot\cancel{(x+4)}}{(x-2)\cancel{(x+4)}} = \cancel{\frac{A(x+4)}{x-2}} + B$$

0 as $x \to -4$

$$\lim_{x\to -4} \frac{3(-4)}{-4-2} = B \Rightarrow B = \frac{-3\cdot 4}{-6} = 2 \quad \checkmark$$

Works well with simple Linear factors.

Note $f(x) = \frac{p(x)}{q(x)}$, $p(x), q(x)$: polynomials

$\deg p \geqslant \deg q$

$\Rightarrow$ use long division to divide $p(x)$ by $q(x)$ to extract integral part + rational function with deg(numerator) < < deg(denominator).

Ex $$f(x) = \frac{2x^3+11x^2+28x+33}{x^2-x+6}$$

deg(numerator) = 3 > 2 deg(denominator)

Need to divide $2x^3+11x^2+28x+33$ by x^2-x+6

$$\begin{array}{r|l} & \quad 2x+13 \\ \hline x^2-x+6 & 2x^3+11x^2+28x+33 \\ - & 2x^3-2x^2+12x \\ \hline & \qquad 13x^2+16x+33 \\ - & \qquad 13x^2-13x+78 \\ \hline & \qquad\qquad 29x-45 \end{array}$$

$$\therefore \quad \frac{2x^3+11x^2+28x+33}{x^2-x+6} = 2x+13+\frac{29x-45}{x^2-x+6}$$

use partial fractions

$x^2 - x + 6$

$$x_{1,2} = \frac{1 \pm \sqrt{1 - 4 \cdot 6}}{2} \quad : \text{ imag. roots}$$

$$\Rightarrow x^2 - x + 6 = (x^2 - x + 1) + 5 = (x-1)^2 + 5 > 0$$

no real roots

this is irreducible form

we stop here

Note We can apply partial fraction decomposition only to ratio of polynomials

$$f(x) = \frac{p(x)}{q(x)}, \quad \text{when } \deg(p) < \deg q$$

Ex

$$\frac{e^{2x}(x^2+2)}{(x-3)(x-7)(x+4)}$$

$$\frac{x^2+2}{(x-3)(x-7)(x+4)} = \frac{A}{x-3} + \frac{B}{x-7} + \frac{C}{x+4}$$

1st degree polynom.

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Ex $$\frac{3x+7}{\underbrace{(x^2+2x+4)}_{\text{has no real roots}}(x-5)^2} = \frac{Ax+B}{x^2+2x+4} +$$

2nd degree polynomial

$$+ \frac{C}{x-5} + \frac{D}{(x-5)^2}$$

1st degree polynomial

In partial fraction decomposition

$$+ \frac{\text{numer}}{\text{denomin.}} + \dots$$

$$\deg(\text{numer}) = \deg(\text{denom.}) - 1$$

8.5 Other Integration Strategies

Not all integrals can be evaluated in terms of familiar functions (in the closed form)

Ex $\int e^{x^2}dx$, $\int \sin(x^2)\,dx$

$\int \frac{\sin x}{x}dx$, $\int \frac{e^{-x}}{x}dx$

$\int \ln(\ln x)\,dx$

Available strategies

1. Table of Integrals (see back of the book)

include list of analytically evaluated integrals, they also include reduction formulas.

2. Computer algebra systems (symbolic computation)

Matlab, Maple, Mathematica

3. Numerical Methods

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} c_i f(x_i)$$

Ex Evaluate $\int \frac{dx}{x\sqrt{2x-9}} =$

Method I : substitution $u^2 = 2x-9$

$$\cancel{2}u\,du = \cancel{2}\,dx$$

$$dx = u\,du$$

$$2x = u^2+9$$

$$x = \frac{1}{2}(u^2+9)$$

$$= \int \frac{\cancel{x}\, du}{\frac{1}{2}(u^2+9)\cdot \cancel{x}} = 2\int \frac{du}{u^2+9} =$$

$$a=3$$

$$= \frac{2}{3}\tan^{-1}\frac{u}{3} + C = \frac{2}{3}\tan^{-1}\frac{\sqrt{2x-9}}{3} + C$$

<u>Method II</u>: using table integral

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}}\tan^{-1}\sqrt{\frac{ax-b}{b}} + C, \quad b>0$$

We have $a=2$, $b=9$.

$$\therefore \int \frac{dx}{x\sqrt{2x-9}} = \frac{2}{\sqrt{9}}\tan^{-1}\sqrt{\frac{2x-9}{9}} + C =$$

$$= \frac{2}{3}\tan^{-1}\frac{\sqrt{2x-9}}{3} + C$$

<u>Ex</u> Evaluate $\int \sqrt{x^2+6x}\, dx =$

Table will have $\int \sqrt{x^2 \pm a^2}\, dx$

complete = square

$$\int \sqrt{x^2 + \overset{2}{2}\cdot\overset{b}{3}\overset{a}{x} + 3^2 - 3^2}\, dx =$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= \int \sqrt{(x+3)^2 - 9}\, dx$$

$$= \begin{vmatrix} u = x+3 \\ du = dx \end{vmatrix} = \int \sqrt{u^2 - 9}\, du \;=$$

$a=3$

Table integral: $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} -$

$- \frac{a^2}{2} \ln\left| x + \sqrt{x^2-a^2} \right| + C$

$$= \frac{u}{2}\sqrt{u^2-3^2} - \frac{3^2}{2} \ln\left| u + \sqrt{u^2-3^2} \right| + C =$$

$$= \frac{x+3}{2}\sqrt{(x+3)^2-9} - \frac{9}{2} \ln\left| x+3 + \sqrt{(x+3)^2-9} \right| + C$$

Ex $\int_0^{\pi} \frac{dx}{1+\sin x} =$

area of region bounded by $y = \frac{1}{1+\sin x}$, x-axis between $x=0$ & $x=\pi$

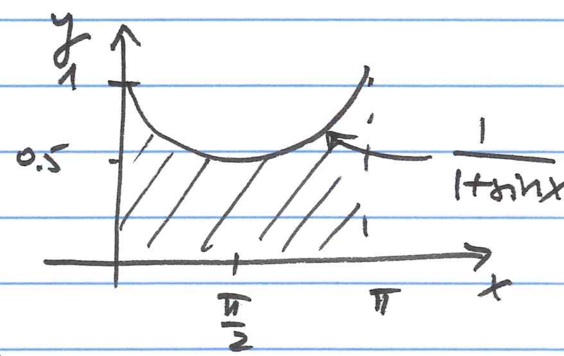


Table integral: $\int \frac{dx}{1+\sin ax} = -\frac{1}{a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

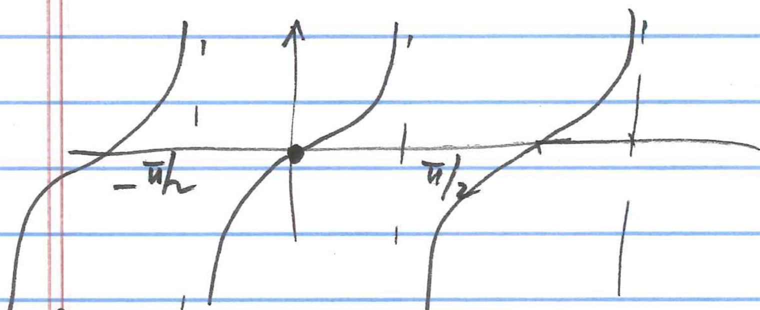
We have $a=1$

$$= -\frac{1}{1} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\Big|_{x=0}^{x=\pi} = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\Big|_0^{\pi} =$$

$$= -\tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) + \tan\left(\frac{\pi}{4}\right) =$$

$$= \underbrace{-\tan\left(-\frac{\pi}{4}\right)} + \tan\left(\frac{\pi}{4}\right) \boxed{=}$$

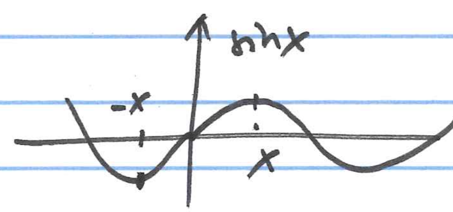
$\tan t$ is odd function $\Rightarrow \tan(-t) = -\tan(t)$



odd function $y = y(x)$

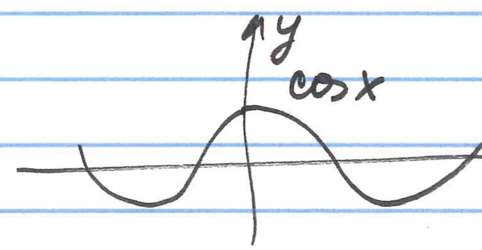
$$y(-x) = -y(x)$$

odd function is symmetric wrt origin



even function $y = y(x)$

$$y(-x) = y(x)$$



even function is symmetric wrt y-axis

$$\int_{-a}^{a} \underset{\text{odd}}{f(x)}\, dx = 0 \qquad \int_{-a}^{a} \underset{\text{even}}{f(x)}\, dx = 2\int_{0}^{a} f(x)\, dx$$

$$\boxed{=} \tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right) = 2\tan\frac{\pi}{4} = 2 \cdot 1 = 2$$

Ex Evaluate $\int \frac{dx}{\sqrt{e^x+1}}$

A Computer algebra system gives the answer:

$$\int \frac{dx}{\sqrt{e^x+1}} = -2\tanh^{-1}\left(\sqrt{e^x+1}\right) + C$$

inverse hyperbolic tangent

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

hyperbolic cosine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

hyperbolic sine

Aside

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$e^{ix} = \cos x + i \sin x$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^x}$$

hyperbolic tangent

Let $z = \tanh x \Rightarrow x = \tanh^{-1} z$

$$z = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$z = \frac{e^{2x} - 1}{e^{2x} + 1}$$: need to solve for e^{2x}

$$z(e^{2x}+1)=e^{2x}-1$$

$$z\cdot e^{2x}+z=e^{2x}-1$$

$$z\cdot e^{2x}-e^{2x}=-z-1$$

$$e^{2x}(z-1)=-(1+z)$$

$$\boxed{e^{2x}=\frac{1+z}{1-z}} \qquad \ln \qquad \ln e^{2x}=2x$$

$$2x=\ln\frac{1+z}{1-z}$$

$$x=\frac{1}{2}\ln\frac{1+z}{1-z}$$

$\shortparallel$

$\tanh^{-1}z$

$$\therefore \quad \boxed{\tanh^{-1}z=\frac{1}{2}\ln\frac{1+z}{1-z}}$$

similarly

$$\boxed{\tan^{-1}z=\frac{1}{2i}\ln\frac{1+iz}{1-iz}}$$

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Hyperbolic functions

$$\boxed{\cosh x = \frac{e^x + e^{-x}}{2} \qquad \cosh ax = \frac{e^{ax} + e^{-ax}}{2}}$$

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2} \qquad \sinh ax = \frac{e^{ax} - e^{-ax}}{2}}$$

$$\frac{d}{dx}\cosh ax = \frac{a e^{ax} - a e^{-ax}}{2} = a\,\frac{e^{ax} - e^{-ax}}{2}$$

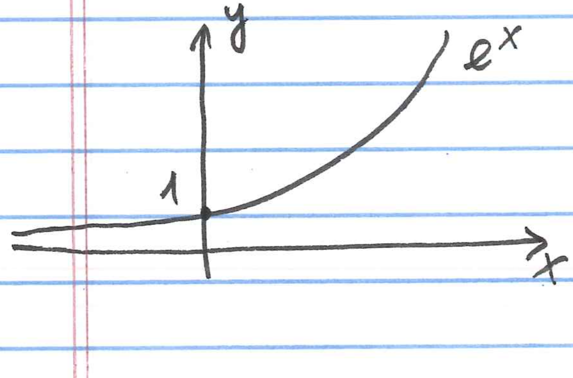
$$= a \sinh ax$$

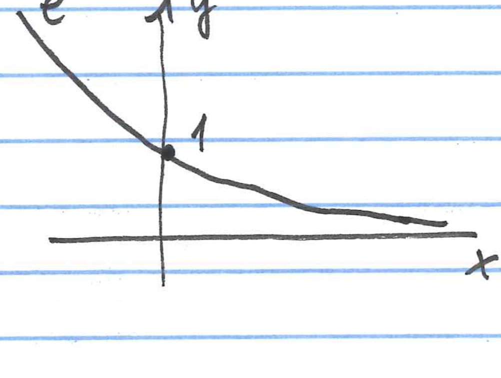
$$\frac{d}{dx}\sinh ax = \frac{a e^{ax} + a e^{-ax}}{2} = a\,\frac{e^{ax} + e^{-ax}}{2}$$

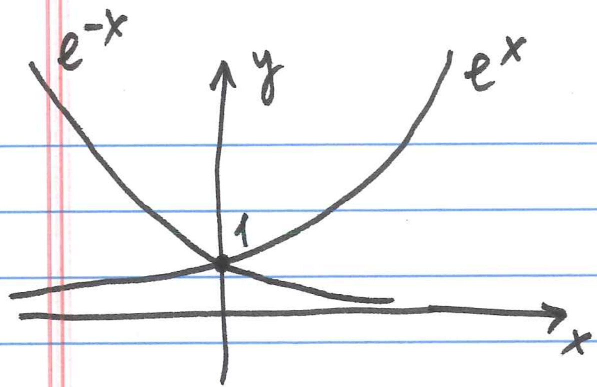
$$= a \cosh ax$$

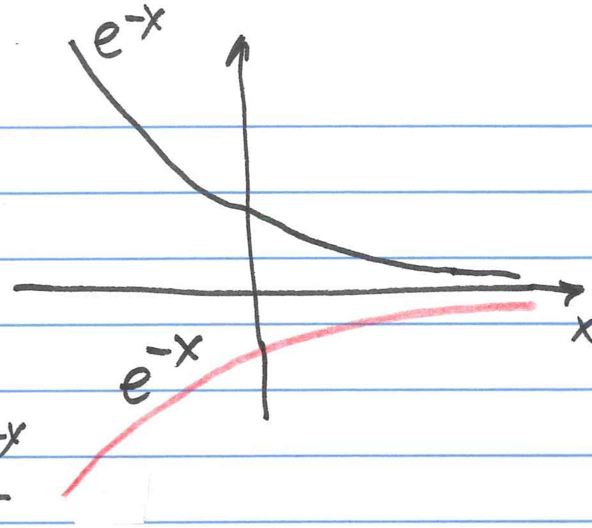
$$\boxed{\frac{d}{dx}\cosh ax = a \sinh ax} \qquad \boxed{\frac{d}{dx}\sinh ax = a \cosh ax}$$

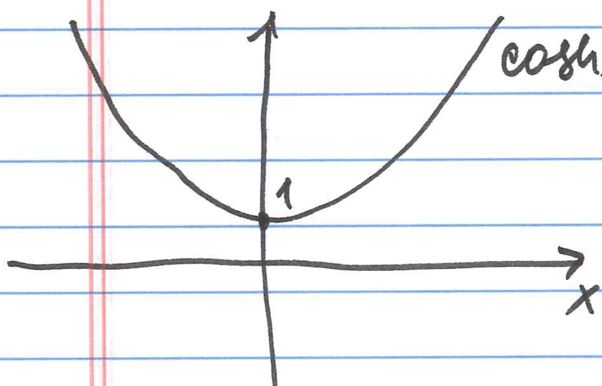
$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$



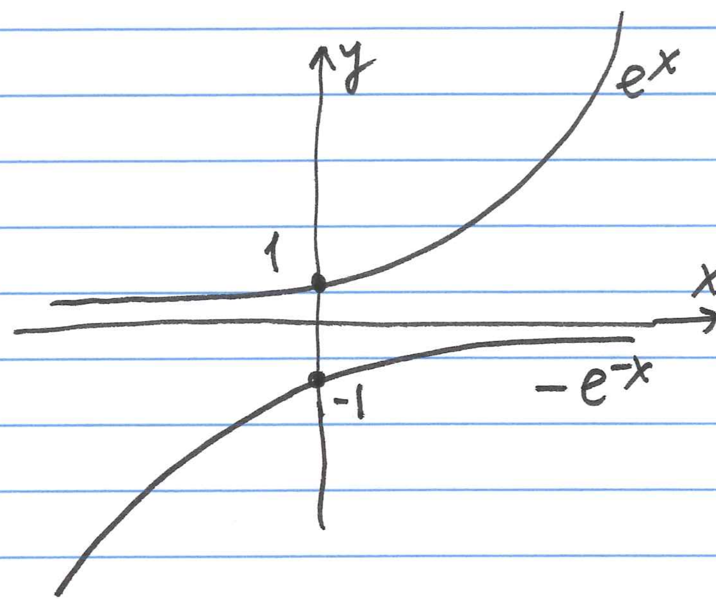




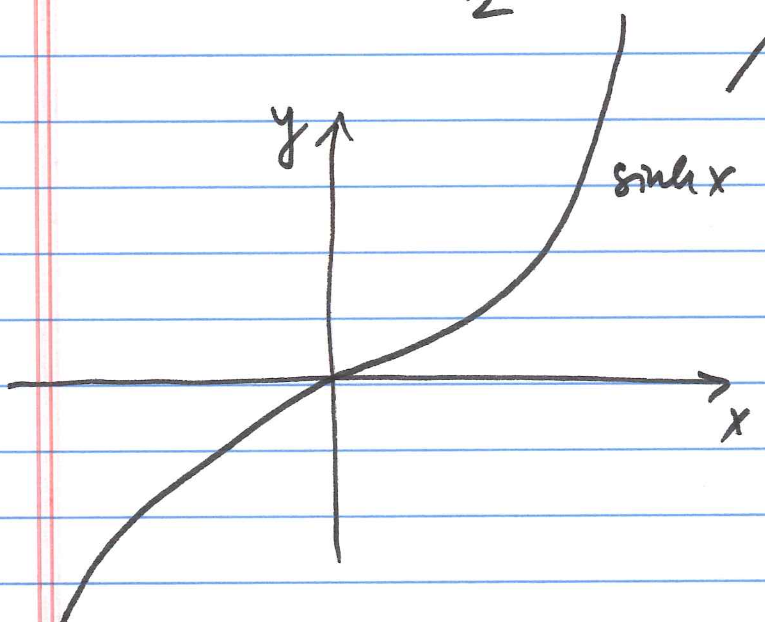




$$\cosh x = \frac{e^x + e^{-x}}{2}$$



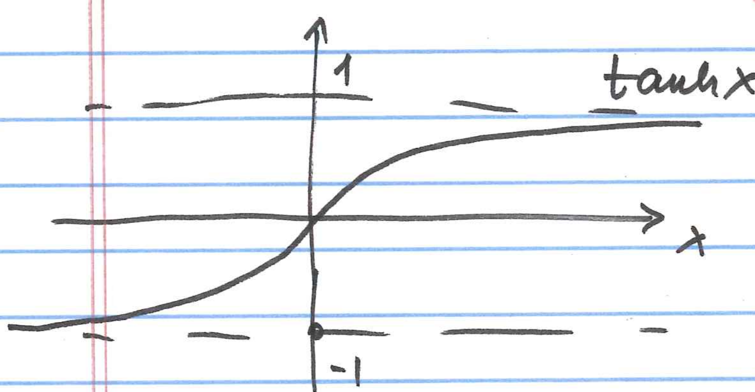
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{2} \cdot \frac{2}{e^x + e^{-x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - \overbrace{e^{-x}}^{\to 0}}{e^x + \underbrace{e^{-x}}_{\to 0}} = 1$$

$$\lim_{x\to-\infty} \tanh x = \lim_{x\to\infty} \frac{\overbrace{e^x}^{\to 0} - e^{-x}}{\underbrace{e^x}_{\to 0} + e^{-x}} = -1$$



Back to example with

(1) $$\int \frac{dx}{\sqrt{e^x+1}} = \boxed{-2\tanh^{-1}\left(\sqrt{e^x+1}\right) + C}$$

computed using symbolic integration

Make substitution

$$\int \frac{dx}{\sqrt{e^x+1}} = \left| \begin{array}{l} u = e^x \\ du = \overbrace{e^x}^{=u} dx \\ dx = \frac{du}{u} \end{array} \right| = \int \frac{du}{u\sqrt{u+1}}$$

Some computer algebra system may give you:

$$\int \frac{du}{u\sqrt{u+1}} = \ln(\sqrt{1+u}-1) - \ln(\sqrt{1+u}+1) + C$$

$$= \ln\left(\frac{\sqrt{1+u}-1}{\sqrt{1+u}+1}\right) + C$$

But the correct answer is

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \ b>0$$

it should be

$$\therefore \int \frac{du}{u\sqrt{u+1}} = \ln\left|\frac{\sqrt{1+u}-1}{\sqrt{1+u}+1}\right| + C =$$

(2) $$= \boxed{\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C}$$

We can show that (1) and (2) are the same by using identity we derived last time

(3) $$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$$

$$\int \frac{dx}{\sqrt{e^x+1}} \stackrel{(1)}{=} -2\tanh^{-1}(\underbrace{\sqrt{e^x+1}}_{z}) + C \stackrel{(3)}{=}$$

$$= -2 \cdot \frac{1}{2} \ln \frac{1+\sqrt{e^x+1}}{1-\sqrt{e^x+1}} + C =$$

$$= -\ln \frac{1+\sqrt{e^x+1}}{1-\sqrt{e^x+1}} + C = \ln \left(\frac{1+\sqrt{e^x+1}}{1-\sqrt{e^x+1}} \right)^{-1} + C =$$

$$a \ln x = \ln x^a$$

$$= \ln \frac{1-\sqrt{e^x+1}}{1+\sqrt{e^x+1}} + C \quad ✓$$

8.6 Numerical Integration

Evaluates $\int_a^b f(x)\,dx$ approximately

x: exact value

c: approximation

absolute error $= |x-c|$

relative error $= \dfrac{|x-c|}{|x|}$, $x \neq 0$

percentage error = relative error × 100%

Ex $e = 2.718281828459046$: exact

Let $c = 2.71$

abs. error $= |e-c| = 0.008281828459046$

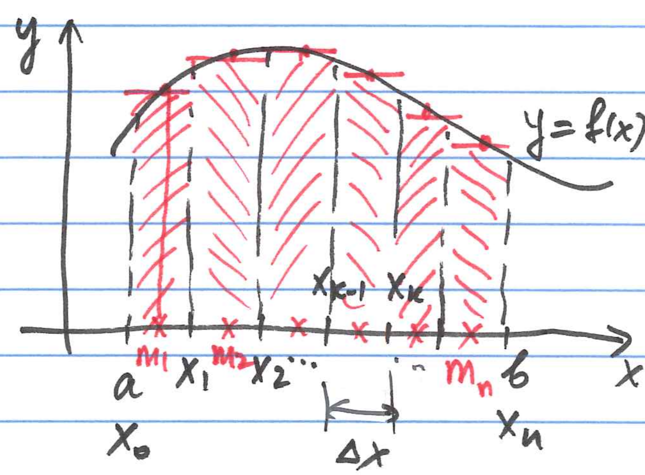
rel. error $= \dfrac{|e-c|}{|e|} = \underbrace{0.0030\ \ 4671}4425391$

≈ 0.0030467

percentage error is 0.30467%

Midpoint Rule

Partition $[a,b]$ into n subintervals of the same length $\Delta x = \dfrac{b-a}{n}$



$n+1$ grid points $x_0, x_1, \dots, x_n$

$x_k = a + \Delta x \cdot k, \quad k = 0, \dots, n$

$[x_{k-1}, x_k]$: k^{th} subinterval $\qquad m_k = \dfrac{x_{k-1} + x_k}{2}$

$$\int_a^b f(x)\,dx \approx M(n) = f(m_1)\Delta x + f(m_2)\Delta x + \dots$$
$$\dots + f(m_n)\Delta x$$

$$M(n) = \sum_{k=1}^{n} f\Big(\underbrace{\frac{x_{k-1}+x_k}{2}}_{m_k}\Big)\,\Delta x$$

$$\int_a^b f(x)\,dx = M(n) + E_{mid}$$

One can show

$$E_{mid} = \frac{f''(\xi)}{24}(b-a)(\Delta x)^2 \quad : \text{ error}$$

ξ is some pt on $[a, b]$

$$|E_{Mid}| \le \frac{|f''(\xi)|}{24}(b-a)(\Delta x)^2$$

let $|f''(\xi)| \le k$
const

$$\Rightarrow \text{error } |E_{Mid}| \le \frac{k}{24}(b-a)(\Delta x)^2$$

2nd order accurate

$$\text{error}\big|_{\Delta x} \sim C\cdot(\Delta x)^2$$

$$\text{error}_{\frac{\Delta x}{2}} \rightarrow C\cdot\left(\tfrac{1}{2}\right)^2(\Delta x)^2 = \tfrac{1}{4}\,\text{error}_{\Delta x}$$

Note as $\Delta x \downarrow \Rightarrow$ error also $\downarrow$