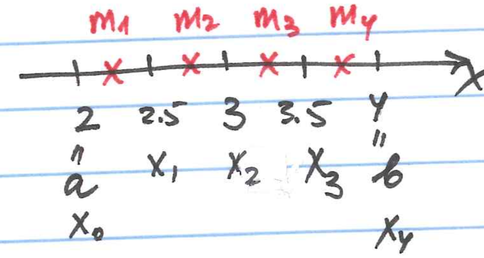
2/18/2014

Ex $\displaystyle\int_2^4 x^2\,dx$ evaluate using Midpoint Rule with $n=4$ and $n=8$.

$\underline{\underline{n=4}}$



$$\int_2^4 x^2\,dx \approx f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + f(m_4)\Delta x =$$

$a=2,\ b=4$

$$f(x)=x^2, \quad \Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$$

$x_0=2,\ x_1=2.5,\ x_2=3,\ x_3=3.5,\ x_4=4$

$m_1=2.25,\ m_2=2.75,\ m_3=3.25,\ m_4=3.75$

$$M(4) = \left(f(m_1)+f(m_2)+f(m_3)+f(m_4)\right)\Delta x =$$

$$= \left(m_1^2+m_2^2+m_3^2+m_4^2\right)\Delta x = \left(2.25^2+2.75^2+3.25^2+3.75^2\right)\cdot\frac{1}{2} = \boxed{18.625}$$

Exact $\displaystyle\int_2^4 x^2\,dx = \frac{x^3}{3}\Big|_2^4 = \boxed{\frac{56}{3}}$

$$\text{abs. error} = \left|\frac{56}{3} - 18.625\right| = \boxed{0.041666\ldots} \approx 0.0417$$

$$\text{rel. error} = \left|\frac{\frac{56}{3}-18.625}{\frac{56}{3}}\right| \approx 0.0022321\ldots \approx 0.22321\%$$

$n=8$ $\quad M(8) = 18.65625$

$$\text{abs. error} = \left| \frac{56}{3} - 18.65625 \right| = \boxed{0.0104166\ldots}$$

$\sim \frac{1}{4}\ \text{abs. er}_{n=4}$

$$\Delta x \to \frac{\Delta x}{2} \Rightarrow er_{\frac{\Delta x}{2}} \sim \frac{1}{4} er_{\Delta x}$$

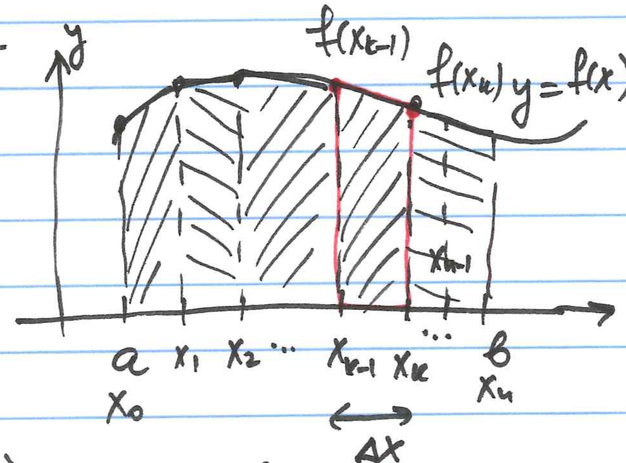
$$\text{rel. error} = \left| \frac{\frac{56}{3} - 18.65625}{\frac{56}{3}} \right| \approx 5.580357 \times 10^{-4}$$

$$\approx 5.58 \cdot 10^{-2}\,\%$$

Trapezoid Rule

$$A_{\text{▨}} = \frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$



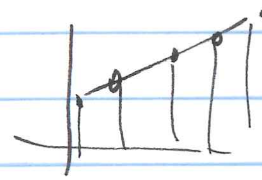
$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x + \frac{1}{2}\left(f(x_1) + f(x_2)\right)\Delta x + \ldots$$

$$+ \frac{1}{2}\left(f(x_{n-2}) + f(x_{n-1})\right)\Delta x + \frac{1}{2}\left(f(x_{n-1}) + f(x_n)\right)\Delta x =$$

$$\boxed{T(n) = \left(\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right)\Delta x}$$

Trapezoid Rule

$$\underbrace{\int_a^b f(x)\,dx}_{\text{exact}} = \underbrace{T(n)}_{\text{approx.}} + \underbrace{\frac{f''(\xi)}{12}(b-a)(\Delta x)^2}_{\text{error}}$$



Let $|f''(\xi)| \le k$, $a \le \xi \le b$

$$\Rightarrow |\text{Error}_T| \le \underbrace{\frac{k}{12}(b-a)}_{C}(\Delta x)^2$$

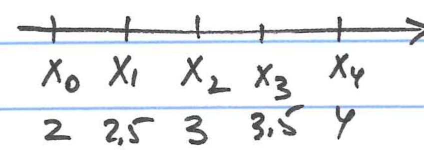
Note: trapezoid rule is exact for linear functions

$$\text{Error}_T \sim C \cdot (\Delta x)^2$$

$$\Rightarrow \Delta x \to \frac{\Delta x}{2} \Rightarrow \text{error} \to \frac{1}{4}\,\text{error}$$

Ex: Apply Trapezoid Rule w/ $n=4$ to evaluate

$$\int_2^4 x^2\,dx \approx T(4)$$



$$T(4) = \left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right)\Delta x =$$

$$\Delta x = \frac{4-2}{4} = \frac{1}{2}, \quad f(x) = x^2$$

$$= \left(\frac{1}{2}x_0^2 + x_1^2 + x_2^2 + x_3^2 + \frac{1}{2}x_4^2\right)\Delta x =$$

$$= \left(\frac{1}{2}\cdot 2^2 + 2.5^2 + 3^2 + 3.5^2 + \frac{1}{2}4^2\right)\cdot\frac{1}{2} = \boxed{18.75}$$

abs. error $= \left|\frac{56}{3} - 18.75\right| = \boxed{0.0833}$ $\approx$ twice larger than error w/ midpoint Rule

rel. error $= |\text{abs. error}| / |56/3| = 0.00446$ or 0.446%

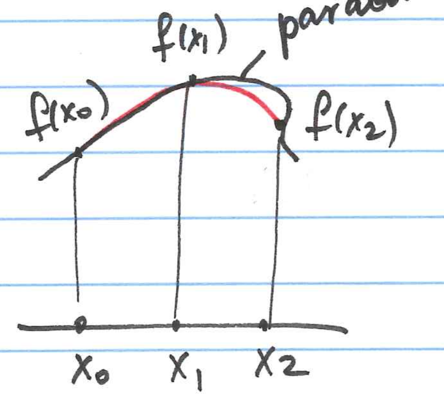
Ex $\int_0^1 x e^{-x}\,dx = 1 - 2e^{-1} = 0.26\ 42111765 7115$

n	$M(n)$	$T(n)$	$E_R\ M(n)$	$E_R\ T(n)$
4	0.26683456	0.259045	0.00259	0.00520
8	0.2648914	0.26293980	0.000650	0.00130
16	0.264403836	0.263915644480	0.000163	0.000325
32	0.264281805	0.2641597404	0.0000407	0.0000814
64	0.2642512900	0.2642207703	0.0000102	0.0000203
128	0.26424366 08	0.264 236031	0.00000254	0.00000509

$\approx \times 2$

$\times \frac{1}{4}$

Simpson's Rule



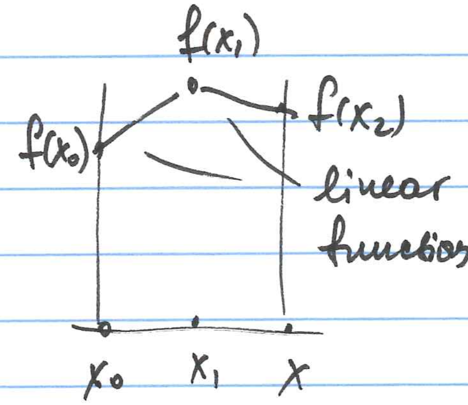
$$\int_a^b f(x)\,dx \approx S(n) =$$

$$= \Big(f(x_0) + 4f(x_1) + 2f(x_2) +$$

$$+ 4f(x_3) + \dots + 2f(x_{n-2}) +$$

$$+ 4f(x_{n-1}) + f(x_n)\Big)\,\Delta x/3$$

n: even



Note $S(2n) = \dfrac{4T(2n) - T(n)}{3} = T(2n) + \dfrac{T(2n) - T(n)}{3}$

gives more accurate approximation than $T(n)$

$$\int_a^b f(x)\,dx = S(n) + \frac{f^{(4)}(\xi)\,(b-a)}{180}(\Delta x)^4$$

exact — approx. — error

$$\text{error} \sim C \cdot (\Delta x)^4$$

$$\Delta x \to \frac{\Delta x}{2} \Rightarrow \text{error} \to \frac{1}{16}\,\text{error}$$

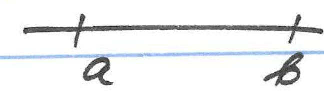
$$\left(\frac{1}{2}\right)^4$$

2/20/2014

8.7 improper integrals

We know how to treat definite integrals of the form

$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx$$



$[a, b]$: finite interval
$f(x)$ assumes finite values on $[a, b]$

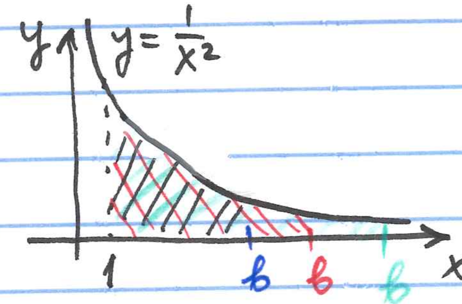
Such integrals are called <u>proper integrals</u>.

<u>Improper integral</u> is when either

— interval of integration is infinite

or / and

— integrand $f(x)$ is unbounded on interval of integration.



<u>Ex</u> $\displaystyle\int_1^b \frac{1}{x^2}\,dx, \quad b>1$

$$= -\frac{1}{x}\Big|_1^b = -\left(\frac{1}{b}-1\right) = 1-\frac{1}{b}$$

as $b \nearrow$, area under curve $y=\frac{1}{x^2}$ also $\nearrow$

Q What happens when $b \to \infty$?

$$\lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1$$

Note: Curve $y = \frac{1}{x^2}$ has infinite length but a region bounded by this curve is finite and has area $= 1$

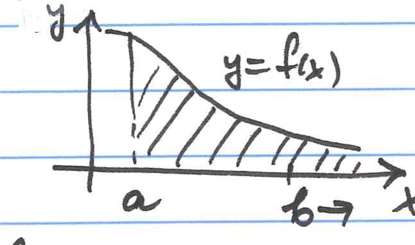
We write this

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1 \quad :$$ improper integral because of ∞ upper limit

1. If $f(x)$ is continuous (CTS) on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{b \to \infty} \int_a^b f(x)\,dx$$

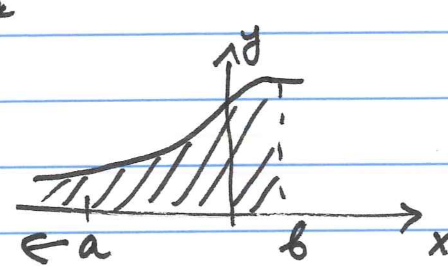
if limit exists.



2. If $f(x)$ is CTS on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx \overset{\text{def}}{=} \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

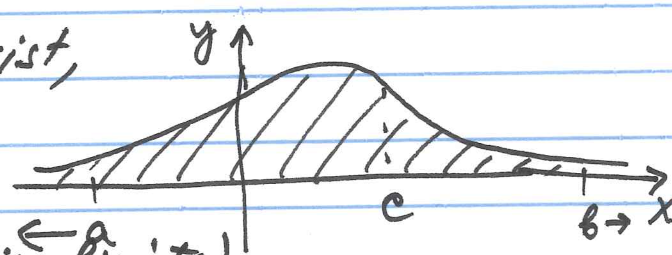
if limit exists.



3. If $f(x)$ is cts on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty} \int_a^c f(x)\,dx + \lim_{b \to \infty} \int_c^b f(x)\,dx$$

provided both limits exist, c is an arbitrary real number.



(limit is finite)

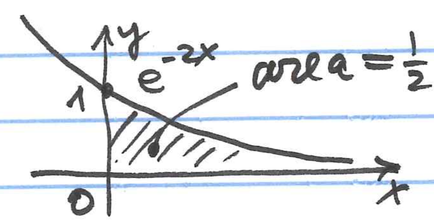
If the limit exists, we say that the improper integral converges. Otherwise, improper integral diverges.

Ex

$$\int_0^{\infty} e^{-2x}\,dx \overset{def}{=} \lim_{b \to \infty} \int_0^b e^{-2x}\,dx =$$

$$= \lim_{b \to \infty} \left(-\tfrac{1}{2}\right) e^{-2x} \Big|_0^b = \lim_{b \to \infty} \left(-\tfrac{1}{2}\right)\left(\underset{\substack{\downarrow \\ 0}}{e^{-2b}} - \underset{\substack{\parallel \\ 1}}{e^0}\right) =$$

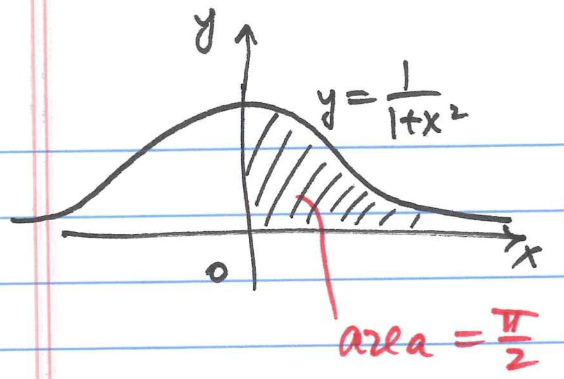
$$= -\tfrac{1}{2}(0-1) = \underset{\text{finite}}{\tfrac{1}{2}} : \text{ improper } \textstyle\int \text{ converges}$$

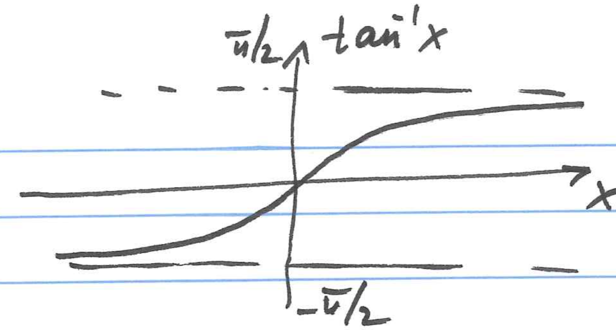


Ex

$$\int_0^{\infty} \frac{dx}{1+x^2} \overset{def}{=} \lim_{b \to \infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b \to \infty} \tan^{-1} x \Big|_0^b =$$

$$= \lim_{b \to \infty} \left(\tan^{-1} b - \overset{0}{\tan^{-1} 0}\right) = \frac{\pi}{2} - 0 = \underset{\text{finite}}{\frac{\pi}{2}} : \text{ improper } \textstyle\int \text{ converges}$$





Ex The family $f(x)=\frac{1}{x^p}$, p is a real #

Q For which p, $\int_1^\infty \frac{dx}{x^p}$ converges?

Assume $p>0$.

$$\int_1^\infty \frac{dx}{x^p} \overset{\text{def}}{=} \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^b =$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(b^{1-p}-1\right)$$

Case 1: $p>1 \Rightarrow 1-p<0$ or $p-1>0$

$$b^{1-p} = \frac{1}{b^{p-1}} \quad {\scriptstyle p-1>0} \Rightarrow \lim_{b\to\infty} b^{1-p} = 0$$

$b\to\infty$

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{1-p} \lim_{b\to\infty} (\underbrace{b^{1-p}}_{\downarrow\, 0} - 1) = \frac{1}{p-1} : \text{converges}$$

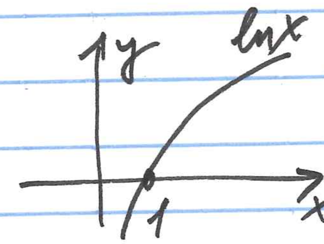
Case 2: $0 < p < 1 \Rightarrow 1-p > 0 \Rightarrow b^{1-p} \xrightarrow[b\to\infty]{} \infty$

$$\therefore \int_1^\infty \frac{dx}{x^p} = \frac{1}{1-p} \lim_{b\to\infty} (\underbrace{b^{1-p}}_{+\infty} - 1) = \infty : \text{diverges}$$

Case 3: $p = 1$

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x \Big|_1^b =$$

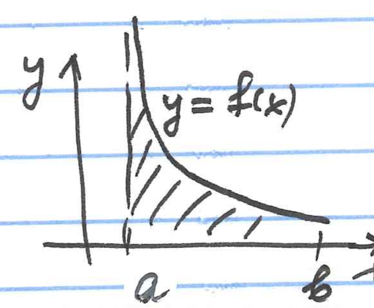
$$= \lim_{b\to\infty} (\ln b - \cancel{\ln 1}^{\,0}) = \infty : \text{diverges}$$



Summary:

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 : \text{converges} \\ \text{diverges} & \text{otherwise } 0 < p \le 1 \end{cases}$$

Unbounded integrands



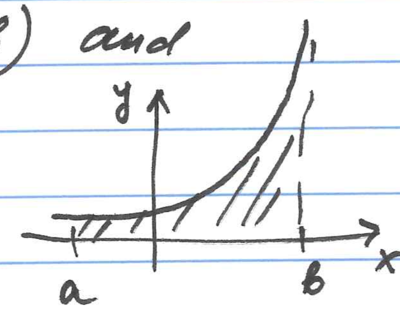
1. Suppose f is cts on $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$. Then

$$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{c\to a^+} \int_c^b f(x)\,dx$$

if limit exists.

2. Suppose f is cts on $[a, b)$ and

$$\lim_{x\to b^-} f(x) = \pm\infty$$



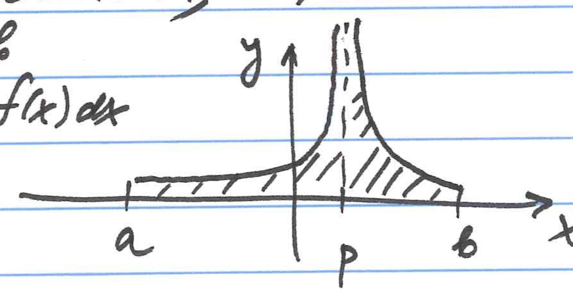
Then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

if limit exists.

3. If $f(x)$ is cts on $[a,b]$ except at pt p where f is unbounded, then

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$



if limits exist.

If the limit is finite, we say that improper integral <u>converges</u>, otherwise it <u>diverges</u>.

Ex $f(x) = \dfrac{1}{\sqrt{9-x^2}}$, $x \in [-3, 3]$.

$-3 \le x \le 3$

$$\int_{-3}^{3} \frac{dx}{\sqrt{9-x^2}} = 2\int_0^3 \frac{dx}{\sqrt{9-x^2}} \overset{def}{=} 2 \lim_{c\to 3^-} \int_0^c \frac{dx}{\sqrt{9-x^2}} \;=$$

even

$f(-x) = f(x)$

$0 \to 3$

Recall $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

or use substit.

$x = 3\sin u$

$a=3$

$$= 2 \lim_{c\to 3^-} \sin^{-1}\frac{x}{3}\Big|_0^c = 2\lim_{c\to 3^-}\left(\sin^{-1}\frac{c}{3} - \cancel{\sin^{-1} 0}^{\,0}\right)$$

$= 2\cdot\left(\frac{\pi}{2} - 0\right) = \pi$: improper $\int$ converges

finite

$c \to 3^- \Rightarrow \frac{c}{3} \to \frac{3}{3} = 1$

$\sin^{-1}(1) = \frac{\pi}{2}$

2/21/2014

8.8 Introduction to Differential Equations

Math 310 - ordinary differential equations
Math 480 - partial differential equations

Natural phenomena involve changes in time or/and in space, and they can be described or modeled by differential equations

$x = f(t)$ $\quad$ t: independent variable
$\quad\quad\quad\quad$ x: dependent variable

$f'(t) = \frac{df}{dt}$: rate of change of f with respect to t

Def a <u>differential equation</u> (DE) is an equation that involves an unknown function and at least one of its derivatives

Ex $\frac{dx}{dt} = x + \sin t$ $\quad\quad$ $x = x(t)$

1st order derivative

this is a DE of the 1st order

Def The <u>order of DE</u> is the order of the highest derivative in the equation

Ex $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$: DE for $y = y(x)$

2nd order derivative

2nd order DE

linear function $y = ax + b$

x has power 1

Def a linear differential equation has the following form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2y}{dx^2} +$$

$$+ a_1(x)\frac{dy}{dx} + a_0(x)y = R(x)$$

this is nth order linear DE

Note in a linear DE, $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ all appear linearly, i.e. as y to the power 1 etc.

1st order linear DE: $a_1(x)\frac{dy}{dx} + a_0(x)y = R(x)$

Ex $\frac{dx}{dt} = x + \sin t$: linear 1^{st} order DE

$\left(\frac{dx}{dt}\right)^1 \quad x^1$

Aside:

$2x+5$: linear

$\ln x + x$: nonlinear

$x^2 + e^x$: nonlinear

Ex $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \underbrace{e^{2x}}_{R(x)}$: linear 2^{nd} order DE

$a_2 = 1 \qquad a_1 = -3 \qquad a_0 = 2$

Ex $2x\frac{dy}{dx} + e^x y = \underset{R(x)}{5}$: 1^{st} order linear DE

$a_1 = 2x \qquad a_0 = e^x$

Ex $y\frac{dy}{dx} + 5 = e^x$: nonlinear DE 1^{st} order

$$\underbrace{y\frac{dy}{dx}}_{\text{nonlinear term}} = e^x - 5$$

Algebra problem: solve $x^2 = 4$. Solution is $x = \pm 2$. When we substitute $x = \pm 2$ into algebraic equation $x^2 = 4$, i.e. $(\pm 2)^2 = 4$ we get an identity $4 \equiv 4$.

Def To solve a DE, we need to find a function $y = y(x)$ that when substituted into the DE produces an identity for all x from some interval I (and may satisfy other conditions - more later).

Ex $y(x) = Ce^{x^2}$, C is an arbitrary const

$$\frac{dy}{dx} = \underbrace{Ce^{x^2}}_{y} \cdot 2x \Rightarrow \frac{dy}{dx} = 2xy :$$ 1st order DE

and $y(x) = Ce^{x^2}$ is a solution of

$$\frac{dy}{dx} = 2xy$$

Check:

$$\underbrace{Ce^{x^2} \cdot 2x}_{\frac{dy}{dx}} = 2x \cdot \underbrace{Ce^{x^2}}_{y} \quad \checkmark$$

identity for all $-\infty < x < \infty$

$y(x) = Ce^{x^2}$: one-parameter family of solutions

Note 1st order DEs have general solutions that depend on one arbitrary constant

n^{th} order DEs have general solutions that depend on n arbitrary constants

Ex Population Model

Time rate of change of population size $P(t)$ is proportional to population size itself.

$P(t)$: population size

$$\underbrace{\frac{dP}{dt}}_{\text{time rate of change of } P} = kP$$

$k>0$: const

1st order DE

proportional

Solution: $P(t) = Ce^{kt}$, C is an arbitrary const

Check:

$$\frac{dP}{dt} = kP$$

$$\underbrace{C \cdot k \cdot e^{kt}}_{\frac{dP}{dt}} = k \cdot \underbrace{Ce^{kt}}_{P} \quad \checkmark$$

$\frac{dt}{P}$ | $\frac{dP}{dt} = kP$: separable DE

$P \neq 0$ $\quad \underbrace{\frac{dP}{P}}_{\text{function of } P} = \underbrace{k\,dt}_{\text{function of } t}$

$$\int \frac{dP}{P} = \int k\,dt$$

exp / $\ln|P| = kt + \tilde{C}$

$$e^{\ln|P|} = e^{kt+\tilde{C}}$$

$$e^{\ln|P|} = e^{kt} \cdot e^{\tilde{C}}$$

$$|P| = e^{kt} \cdot e^{\tilde{C}}$$

$P(t) = Ce^{kt}$: the general solution of

$$\frac{dP}{dt} = kP$$

The above method is called <u>separation of variables</u>.

2/24/2014

Intro to Differential Equations (Cont'd)

Def A differential equation $\frac{dy}{dt} = f(y)$ has a constant solution $y(t) \equiv c$ if and only if $f(c) = 0$. In this case, the constant c is called an equilibrium solution of DE $y' = f(y)$.

Ex $y' = \underbrace{y}_{f(y)}$ has one equilibrium solution $y \equiv 0$ or $y \equiv c = 0$

Ex $y' = \underbrace{y^2 - 1}_{f(y)}$ has two equil. solutions $c = \pm 1$ or $y = \pm 1$

Def An equilibrium solution c is stable if $y(t) \to c$ as $t \to \infty$ for all nearby solutions $y(t)$. Otherwise, an equilibrium solution is unstable.

Ex Population model

$$y' = \underbrace{ky}_{f(y)} \qquad k \neq 0$$

has 1 equil. solution $y \equiv 0$

Last time we solved this equation and got the solution

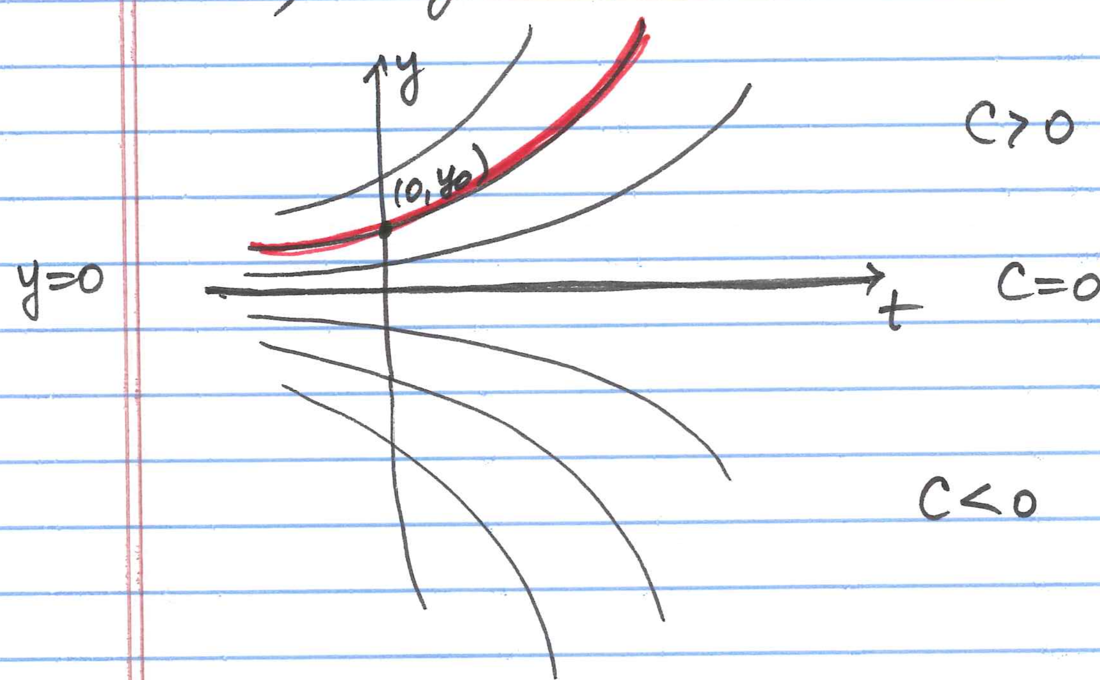
$$y(t) = Ce^{kt}, \quad C \text{ is arbitrary constant}$$

Let $y(0) = y_0$: initial condition

at $t=0$: $\quad y_0 = C\underbrace{e^{k\cdot 0}}_{=1} \Rightarrow C = y_0$

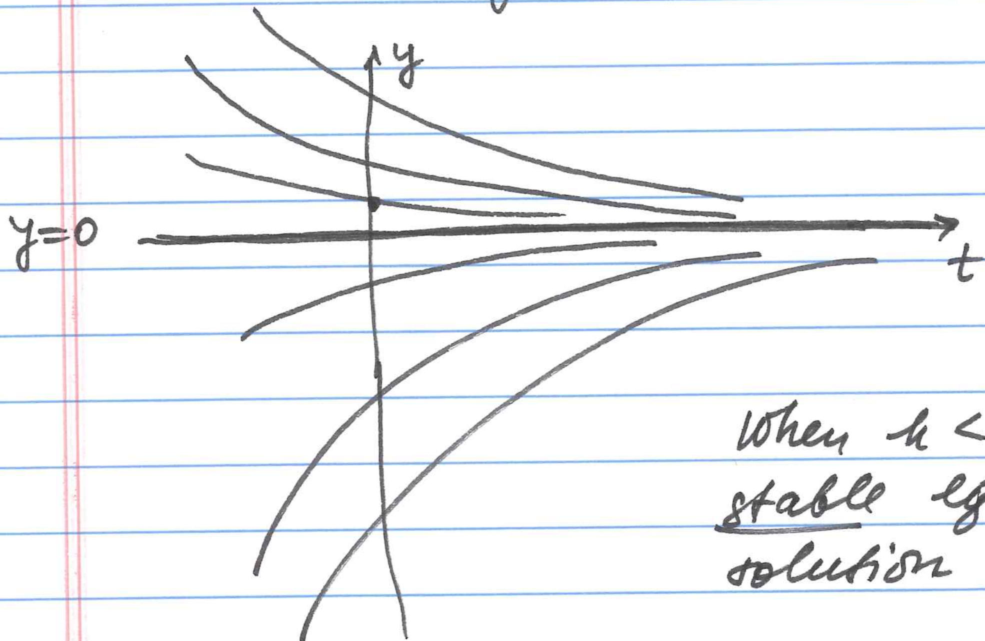
$\therefore \quad y(t) = y_0 e^{kt}$: <u>particular solution</u>

Let $k>0$, $\quad y(t) = Ce^{kt}$



$y=0$ is an unstable equilibrium solution.

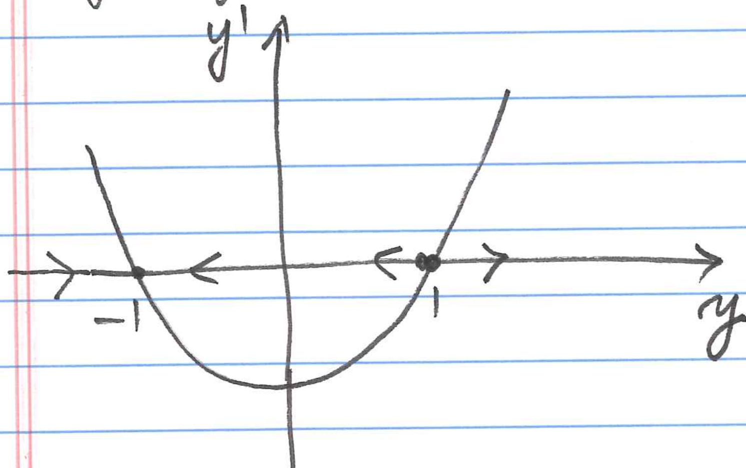
Let $k<0$. $y(t) = Ce^{kt}$



When $k<0$, $y \equiv 0$ is <u>stable</u> equilibrium solution

<u>Ex</u> $y' = y^2 - 1$: $y = \pm 1$

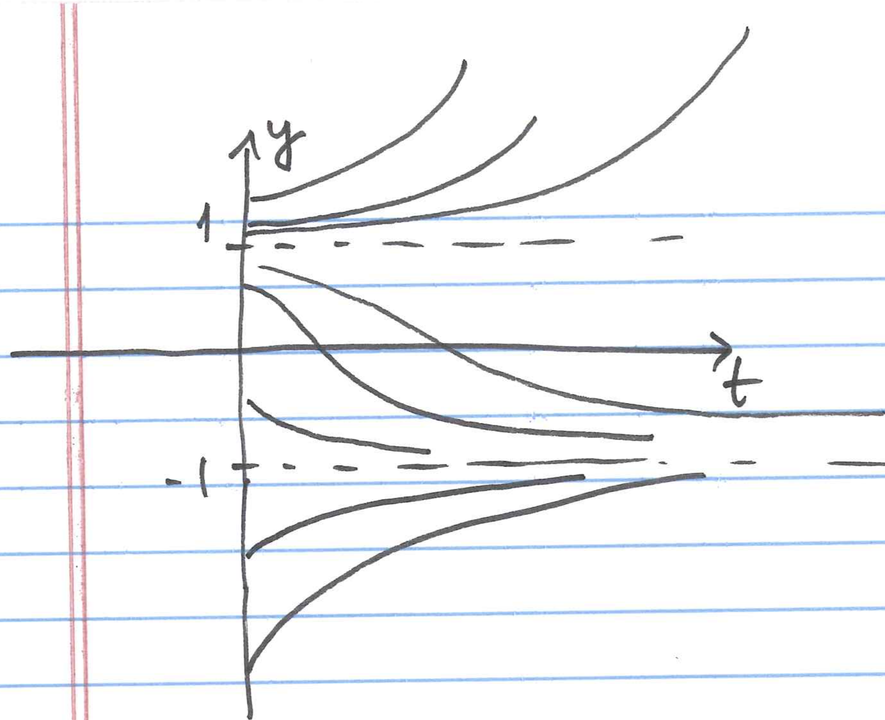
<u>phase plane</u>



phase plane: graphs y' as a function of y

$y=-1$ is stable equil. solution

$y=1$ is unstable equil. solution



$y=1$ is unstable equil. sol^{n}

$y=-1$ is stable equil. sol^{n}

Ex $P(t)$: population size

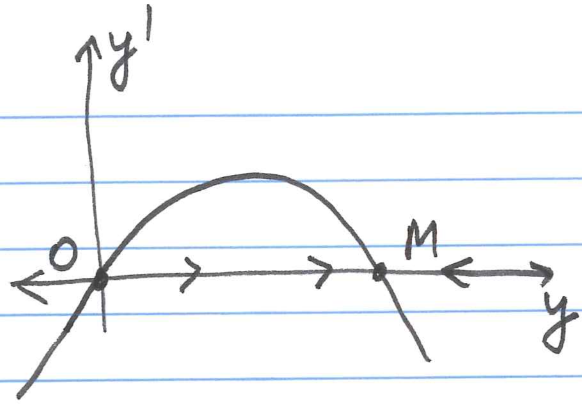
M: population threshold or maximum population size that environment can support or carrying capacity of environment (due to limited resources), $M>0$

$$\frac{dP}{dt} = \underbrace{k(M-P)}_{\text{variable growth rate}} P, \quad k>0 \; : \; \underline{\text{logistic equation}}$$

$f(P) = k(M-P)P$ or $f(y) = k(M-y)y$

$y=0$, $y=M$: two equil. solutions

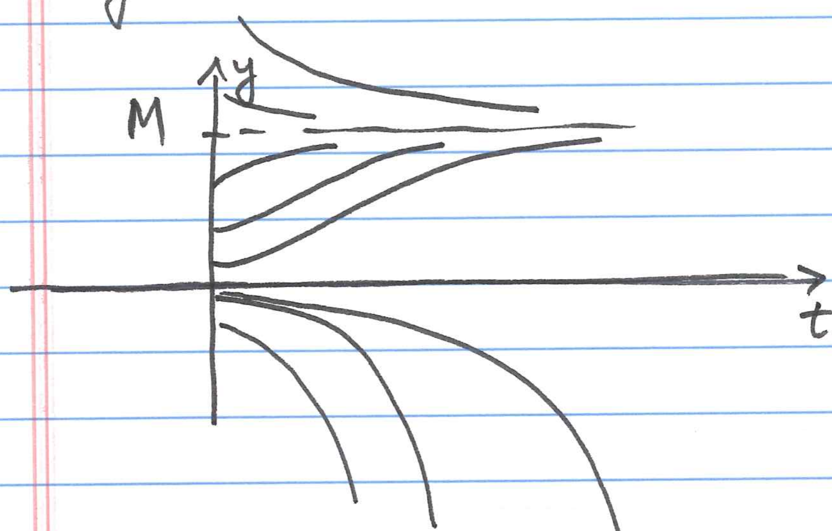
phase plane



$y=0$ is unstable equil. solution

$y=M$ is stable —"—

$y=M$



stable

$y=0$

unstable

Solution of $\frac{dP}{dt} = k(M-P)P$: separable DE

$$\frac{dP}{(M-P)P} = k\,dt$$

$$\int \frac{dP}{(M-P)P} = \int k\,dt$$

$$\frac{1}{(M-P)P} = \frac{1}{M}\left(\frac{1}{M-P} + \frac{1}{P}\right)$$: partial fraction decomposition

after some algebra we can get

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}$$

where $P_0 = P(0)$: initial population size

Q What is $P(t)$ if $P_0 = 0$? $\Rightarrow P(t) = 0$

if $P_0 = M$? $\Rightarrow P(t) = M$ for all t

This confirms that $P(t) = 0$ and $P(t) = M$ are equilibrium solutions.

Q if $0 < P_0 < M$

$\lim_{t\to\infty} P(t)$ - ?

$$\lim_{t\to\infty} \frac{MP_0}{P_0 + \underbrace{(M-P_0)e^{-kMt}}_{\to 0}} = M$$

$\therefore$ $P(t) = M$ is stable equil. solution and $P(t) = 0$ is unstable —||—

This confirms what we knew from stability analysis.

Ex Logistic equation with harvesting

$$\frac{dx}{dt} = kx(M-x) - h \quad , \qquad h>0: \text{ parameter}$$

$h=0$: logistic equation

Ex $\frac{dx}{dt} = x(4-x) - h$ $\qquad k=1$, $M=4$

$x(t)$: population size (measured in hundreds)

$M=4,\ h=0 \Rightarrow \lim_{t\to\infty} x(t) = 4$ (hundred fish)

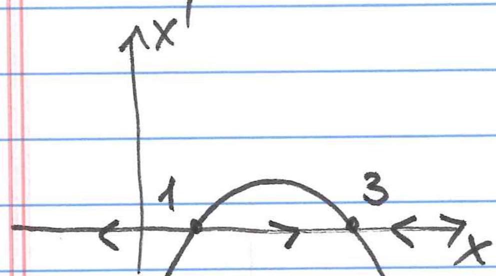
Let $h=3$

$$\frac{dx}{dt} = x(4-x) - 3$$

$$\frac{dx}{dt} = -(x^2-4x+3)$$

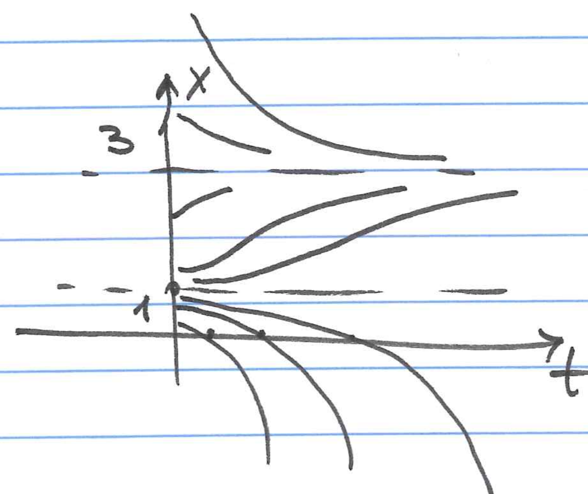
1, 3: roots

$$\frac{dx}{dt} = -(x-1)(x-3)$$



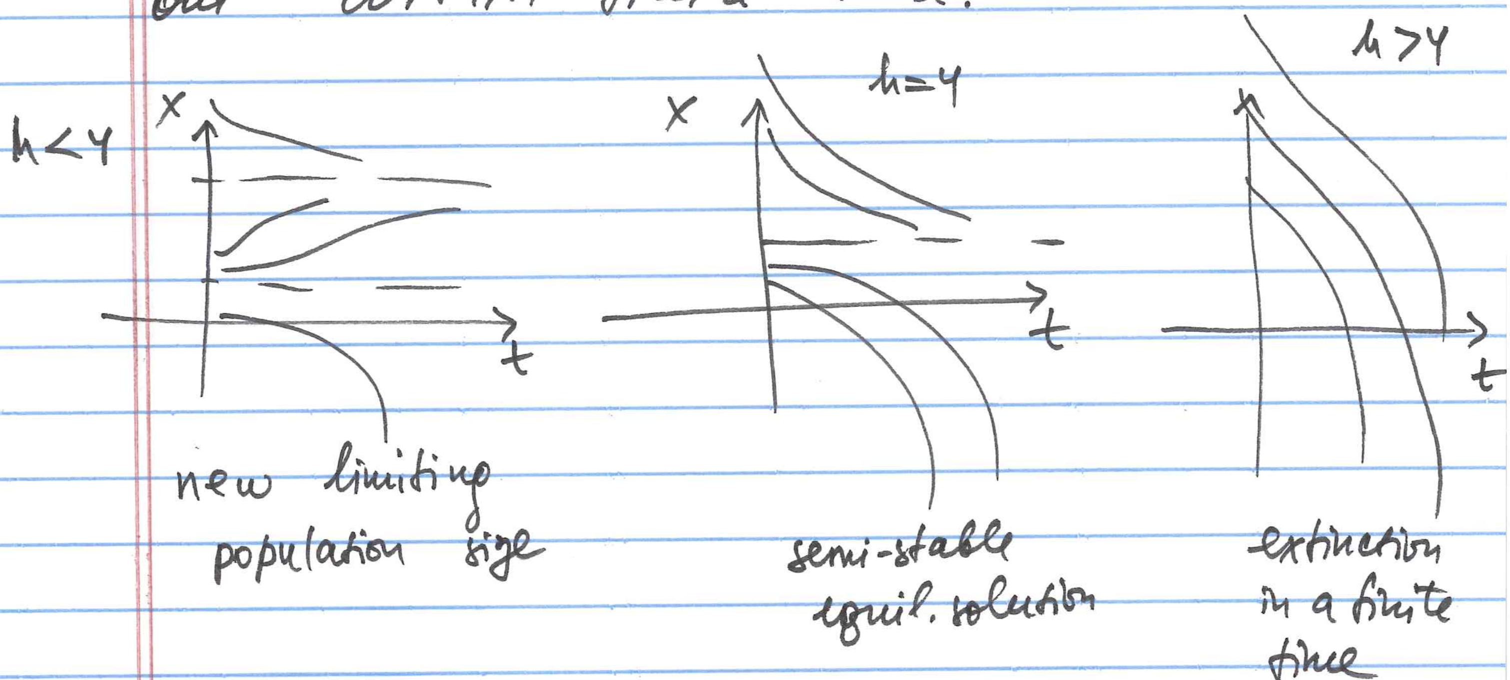
$x=1$: unstable

$x=3$: stable



3 (hundred) fish is a new limiting population size

If lake is stocked initially with more than 100 fish, then $x(t) \to 300$ fish as $t \to \infty$.
If lake is stocked initially w/ fewer than 100 fish, then lake will be "fished out" within finite time.



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8.8 (Cont'd) Slope Fields and Solution Curves

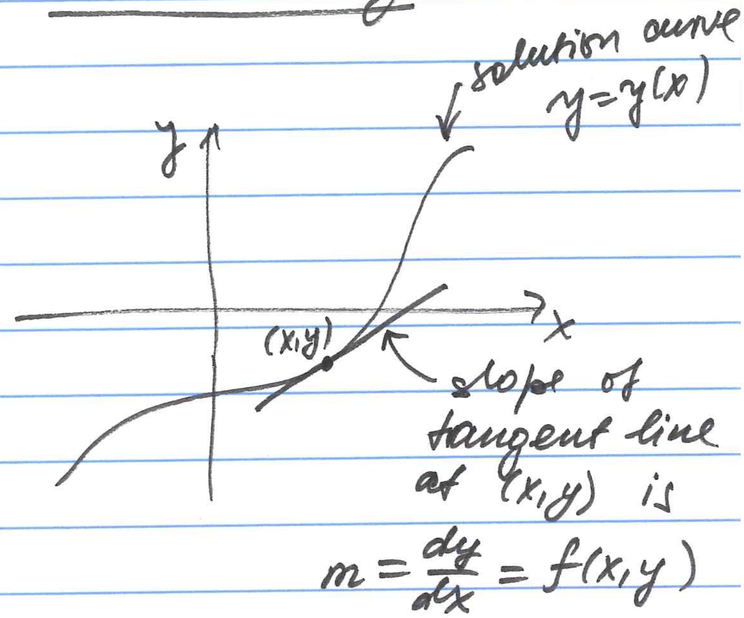
Consider a differential equation of the 1st order

$$\frac{dy}{dx} = f(x,y)$$

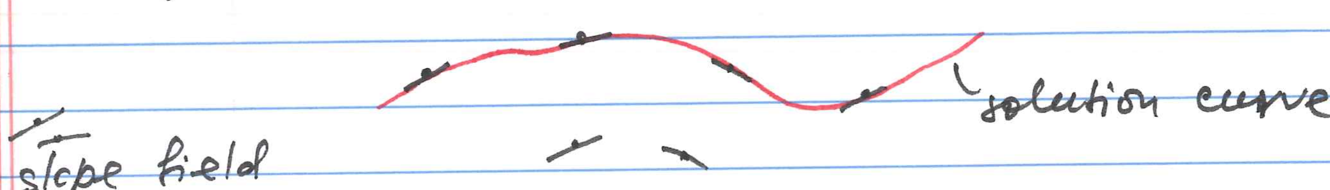
In general, it is not possible to find solution $y(x)$ explicitly. But this solution can be approximated numerically or graphically.

$y = y(x)$: solution

Solution curve is a graph of a solution



At every pt (x,y) we draw a small segment whose slope is $f(x,y)$. The collection of all such segments is slope field or direction field. Solution curve is tangent to these segments at all pts (x,y).



Ex $\dfrac{dy}{dx} = ky, \qquad k = 1$

$\Rightarrow \dfrac{dy}{dx} = \underbrace{y}_{f(x,y)} \Rightarrow$ slope $m\Big|_{(x,y)} = f(x,y) = y$

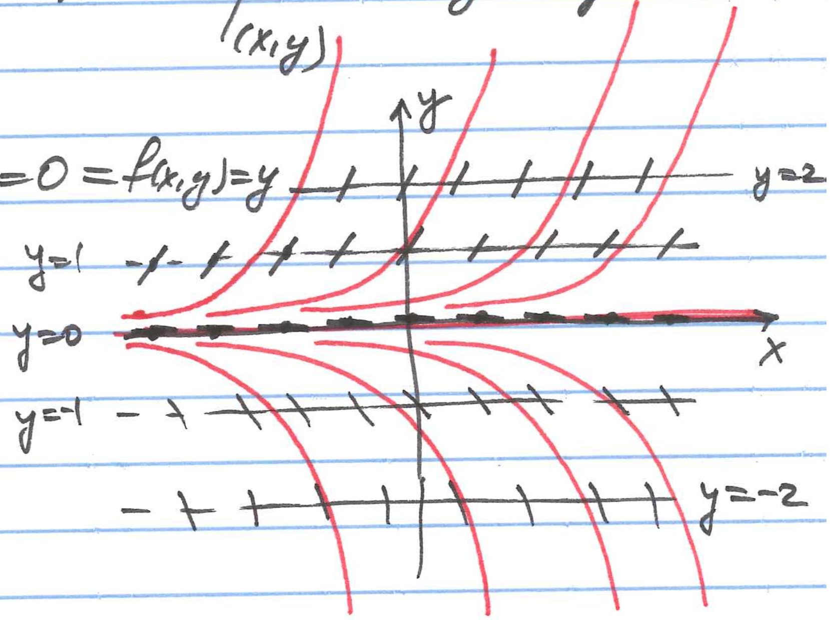
x-axis $\Rightarrow y=0 \Rightarrow m = 0 = f(x,y) = y$

$y=1 \Rightarrow m = y = 1$

$y=2 \Rightarrow m = y = 2$

$y=-1 \Rightarrow m = y = -1$

$y=-2 \Rightarrow m = y = -2$



—: solution curves

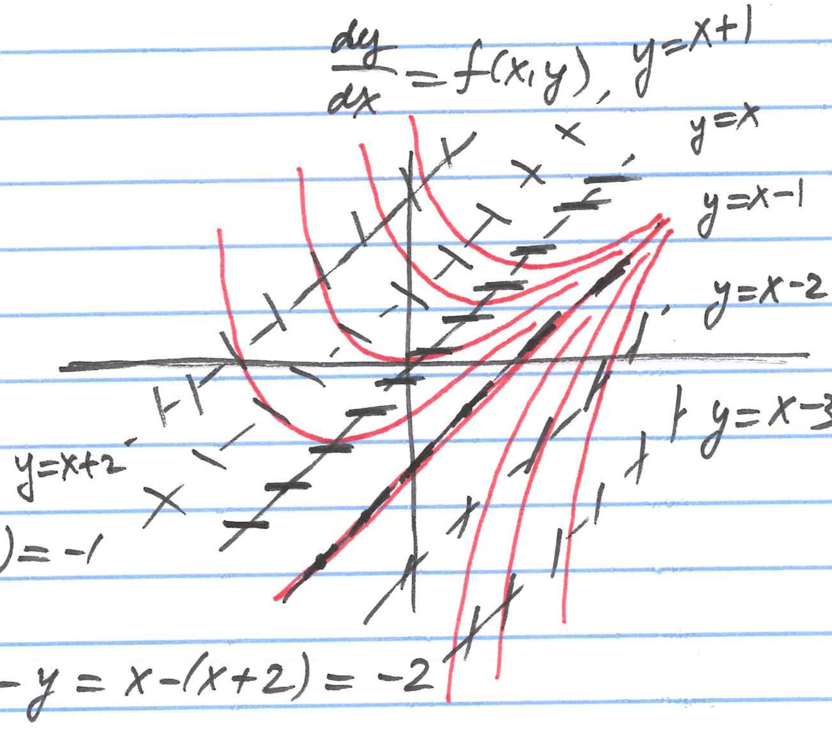
Ex $\dfrac{dy}{dx} = \underbrace{x - y}_{f(x,y) = m}$

Let $y = x$. $\Rightarrow$

$m = x - y = 0$

Let $y = x+1$

$\Rightarrow m = x - y = x - (x+1) = -1$



$y = x+2 \Rightarrow m = x - y = x - (x+2) = -2$

$y = x-1 \Rightarrow m = x - (x-1) = 1$

$y = x - 2 \Rightarrow m = 2$

<u>Note</u>: $y = x - 1$ is a solution of $\frac{dy}{dx} = x - y$

Indeed,
$$\frac{dy}{dx} = 1 \quad , \quad x - y = x - (x-1) = 1$$
$$\Rightarrow \frac{dy}{dx} = x - y \quad \checkmark$$

$$\frac{dy}{dx} = kx(M - x)$$

9. Sequences and Infinite Series

9.1. An Overview

Consider a list of numbers:

$$\{1, 4, 7, 10, 13, 16, \ldots\} \quad : \text{ a sequence}$$
$$\quad +3 \; +3 \; \ldots$$

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\} \text{ or } \{a_n\}_{n=1}^{\infty} \text{ or } \{a_n\}$$

a_n: n^{th} term

n: index, usually $n=0$ or $n=1$

In this example, we can write

$$a_{n+1} = a_n + 3 \; : \quad n = 1, 2, \ldots$$
$$a_1 = 1$$

<u>currence relation</u> or <u>implicit formula</u>

Another way:

$a_1 = 1 = 1 + 3 \cdot 0$

$a_2 = 4 = 1 + 3 \cdot 1$

$a_3 = 7 = 1 + 3 \cdot 2$

$\cdot \; - \; \cdot$

$a_n = 1 + 3 \cdot (n-1) = 3n - 2, \quad n = 1, 2, \ldots$

$a_n = 3n-2, \quad n = 1, 2, \ldots$: explicit formula

Def a sequence $\{a_n\}$ is an ordered list of numbers of the form

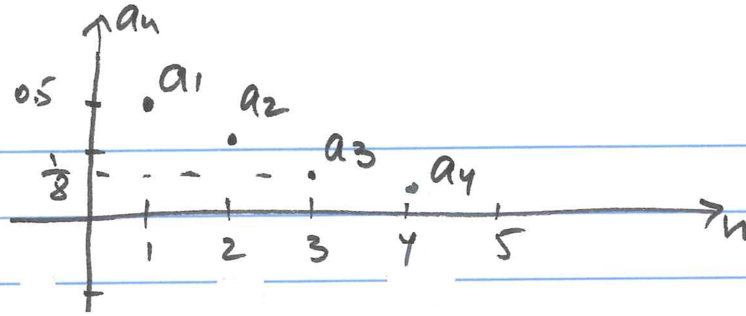
$$\{a_1, a_2, \ldots, a_n, \ldots\}$$

A sequence may be generated by a recurrence formula of the form $a_{n+1} = f(a_n)$ with a_1 given, or by explicit formula for n^{th} term:

$a_n = f(n), \quad n = 1, 2, \ldots$

Ex $a_n = \frac{1}{2^n}$: explicit

$a_1 = \frac{1}{2^1}, \quad a_2 = \underbrace{\frac{1}{2^2}}_{=\frac{1}{4}}, \quad a_3 = \underbrace{\frac{1}{2^3}}_{=\frac{1}{8}}, \ldots$

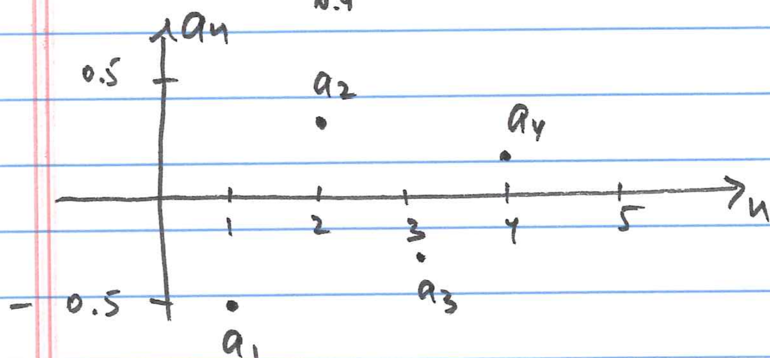


$a_n \to 0$ as $n \nearrow$

Ex $\quad a_n = \dfrac{(-1)^n n}{n^2+1}$

$\left\{ \dfrac{-1\cdot 1}{1^2+1}, \dfrac{2}{2^2+1}, \dfrac{-3}{3^2+1}, \ldots \right\}$

$\left\{ -\dfrac{1}{2}, \underbrace{\dfrac{2}{5}}_{0.4}, \dfrac{-3}{10}, \dfrac{4}{17}, \ldots \right.$



$a_n \to 0$ as $n \nearrow$

Ex $\quad \{b_n\} = \{3, 6, 12, 24, \ldots\}$

$\times 2 \quad \times 2$

$b_{n+1} = 2b_n, \qquad b_0 = 3$

or

$b_n = 3 \cdot 2^n, \quad n = 0, 1, 2, \ldots$