2/27/2014

9.1 Sequences and Series (Cont'd)

Limit of a sequence is the long-term behaviour of a sequence.

Def If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say that $\lim_{n\to\infty} a_n = L$ exists and the sequence converges to L. If the terms of the sequence do not approach a single number as n increases, we say that the sequence has no limit or sequence diverges.

Ex $$\left\{\frac{(-1)^n}{n^2+1}\right\}_{n=1}^{\infty} = \left\{\frac{-1}{1^2+1}, \frac{1}{2^2+1}, \frac{-1}{3^2+1}, \frac{-1}{4^2+1}\ldots\right\}$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0 \ : \ \text{sequence converges}$$

Ex $$\{\cos(n\pi)\}_{n=1}^{\infty}$$

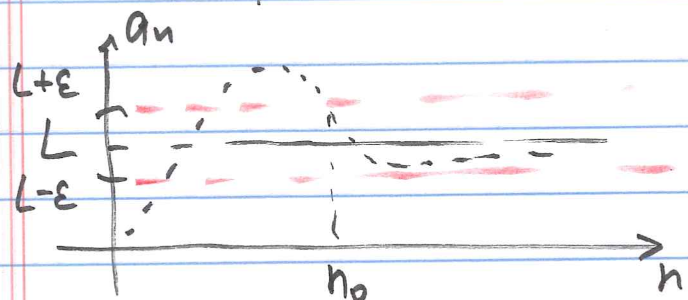
$\{-1, 1, -1, 1, -1, 1, \ldots\}$ no limit

$\therefore \{\cos(n\pi)\}_{n=1}^{\infty}$ diverges a formal def of a limit of a sequence.

Def We say that $\lim_{n\to\infty} a_n = L$ if

for any $\varepsilon > 0$ there exists an index $n_0 \in N$ such that for any $n > n_0$

$$|a_n - L| < \varepsilon \quad \Leftrightarrow L-\varepsilon < a_n < L+\varepsilon$$



in a short form:

$$\forall \varepsilon > 0 \quad \exists n_0 \in N: \quad \forall n > n_0 \quad |a_n - L| < \varepsilon$$

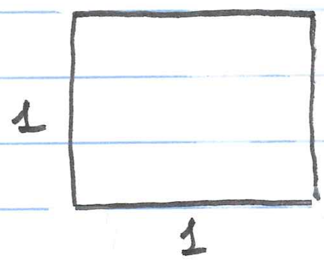
Infinite Series and the Sequences of Partial Sums

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k :$$ infinite series

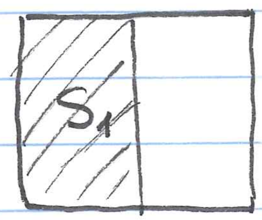
a_k: real numbers

Q What can we say about the sum of an infinite series?

<u>Ex</u> Consider a unit square

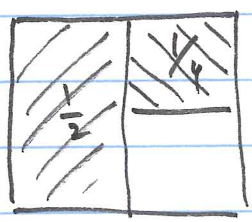


Its area equals 1.

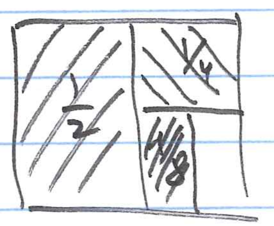


$S_1 = \frac{1}{2}$

S_n: area of shaded region



$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$



$$S_3 = \overset{(4}{\frac{1}{2}} + \underset{\frac{1}{2^2}}{\underset{\|}{\overset{(2}{\frac{1}{4}}}} + \underset{\frac{1}{2^3}}{\underset{\|}{\frac{1}{8}}} = \frac{7}{8}$$

etc.

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n}$$

$$\lim_{n\to\infty} S_n = \underbrace{\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \dots}_{\substack{\text{sum of} \\ \text{infinite series} \\ \sum_{k=1}^{\infty} \frac{1}{2^k}}} = \underset{\substack{| \\ \text{area of} \\ \text{the entire} \\ \text{unit square}}}{1}$$

S_n : partial sum

$\{S_n\}$: sequence of partial sums

$\therefore$ Sum of the infinite series is the limit of sequence of partial sums.

Def Infinite series

Given a set of numbers $\{a_1, a_2, \dots\}$, the sum

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{k=1}^{\infty} a_k$$

is an infinite series. The sequence of its partial sums $\{S_n\}$ has the terms

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$\dots$$

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k, \quad n = 1, 2, \dots$$

If the sequence of partial sums $\{S_n\}$ has a limit L, we say that the infinite series $\sum_{k=1}^{\infty} a_k$ converges to L

and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L$$

If the sequence of partial sums $\{S_n\}$ diverges, we say that $\sum_{k=1}^{\infty} a_k$ also <u>diverges</u>

<u>Ex</u> $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

$$S_1 = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$S_2 = \sum_{k=1}^{2} \frac{1}{k(k+1)} = \overset{(3}{\frac{1}{2}} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \sum_{k=1}^{3} \frac{1}{k(k+1)} = \overset{(6}{\frac{1}{\underset{1\cdot 2}{\underset{\shortparallel}{2}}}} + \overset{(2}{\frac{1}{\underset{2\cdot 3}{\underset{\shortparallel}{6}}}} + \frac{1}{\underset{3\cdot 4}{\underset{\shortparallel}{12}}} = \frac{9}{12} = \frac{3}{4}$$

$$S_4 = \sum_{k=1}^{4} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

$$\cdots$$

$$S_n = \frac{n}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Hence, infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges and has sum 1, ie.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim \underbrace{\frac{n}{n+1}}_{S_n} = 1$$

Ex $\sum_{k=1}^{\infty} (-1)^k \cdot k$

$S_1 = \boxed{-1}$

$S_2 = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 = -1 + 2 = \boxed{1}$

$S_3 = (-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3 = -1 + 2 - 3 = \boxed{-2}$

$S_4 = -1 + 2 - 3 + 4 = \boxed{2}$

$S_5 = -1 + 2 - 3 + 4 - 5 = \boxed{-3}$

$\{S_n\}$ has no limit $\Rightarrow \sum_{k=1}^{\infty} (-1)^k \cdot k$ diverges

9.2 Sequences

Limit of a sequence $\{a_n\}$ as $n \to \infty$ is similar to the limit of a function $f(x)$ as $x \to \infty$ but n assumes integer values.

Given a sequence $\{a_n\}$, we define a function f such that $f(n) = a_n$.

Ex $a_n = \frac{n}{n+1} \Rightarrow f(x) = \frac{x}{x+1}$

We know $\lim_{x\to\infty} f(x) = 1 \Rightarrow \lim_{n\to\infty} \frac{n}{n+1} = 1$ as well

Thm 9.1 Suppose f is a function : $f(n) = a_n$ for any n.
Then if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$

Ex $\{a_n\}_{n=0}^{\infty} = \left\{\frac{5n^2}{n^2+1}\right\}_{n=0}^{\infty}$

$$f(n) = a_n = \frac{5n^2}{n^2+1} \Rightarrow f(x) = \frac{5x^2}{x^2+1}$$

$$\lim_{x\to\infty} f(x) = 5 \Rightarrow \lim a_n = \lim_{n\to\infty} \frac{5n^2}{n^2+1} = 5$$

Thm 9.2 Properties of limits of sequences

Assume $\{a_n\}$ and $\{b_n\}$ are two sequences that have limits A and B, respectively. Then

1. $$\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$$

2. $$\lim_{n\to\infty} c a_n = cA,$$ where c is a real #.

3. $$\lim_{n\to\infty} a_n b_n = A \cdot B$$

4. $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}, \quad B \neq 0$$

2/28/2014

Exam 2: Thursday, March 6, in class

Review: Wednesday, March 5, 4:30 – 5:20
location TBA

Exam will cover sections since Exam 1
up to and including section 9.3.

Ex $a_n = \underbrace{\frac{3n^3}{n^3+1}}_{f(n)}$ $\qquad \{a_n\}_{n=0}^{\infty}$

$$\Rightarrow f(x) = \frac{3x^3}{x^3+1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^3}{x^3+1} = 3 \lim_{x\to\infty} \frac{x^3}{x^3+1} = 3\cdot 1 = 3$$

$$\Rightarrow \lim_{n\to\infty} a_n = 3$$

similar

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3 \lim_{n\to\infty} \frac{n^3}{n^3+1} = 3\cdot 1 = 3$$

#14 §9.2 **Ex** $a_n = n^{1/n}$. Determine whether sequence $\{a_n\}$ converges or diverges.

$$\lim_{n\to\infty} a_n =$$

$$= \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} =$$

$$= e^{\lim_{n\to\infty} \frac{1}{n} \ln n} \equiv e^{L}$$

$$L = \lim_{n\to\infty} \frac{1}{n} \ln n = 0 \cdot \infty = \lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty} \underset{\text{Rule}}{\overset{\text{l'Hop.}}{=}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

$$\therefore \lim_{n\to\infty} n^{\frac{1}{n}} = e^{0} = 1$$

Def a sequence $\{a_n\}$ is <u>nondecreasing</u> if

$$a_{n+1} \geq a_n$$

Note if $a_{n+1} > a_n \Rightarrow \{a_n\}$ is <u>(strictly) increasing</u>

Ex $\{a_n\}_{n=1}^{\infty} = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \{0, \underbrace{1-\tfrac{1}{2}}_{\frac{1}{2}}, \underbrace{1-\tfrac{1}{3}}_{\frac{2}{3}}, \underbrace{1-\tfrac{1}{4}}_{\frac{3}{4}}, \ldots\}$

Def a sequence $\{a_n\}$ is <u>nonincreasing</u> if

$$a_{n+1} \leq a_n$$

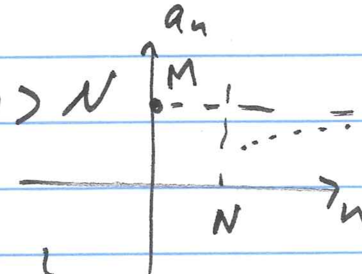
Note if $a_{n+1} < a_n$: $\{a_n\}$ is <u>(strictly) decreasing</u>

Ex $\{a_n\}_{n=1}^{\infty} = \left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} = \{2, 1+\tfrac{1}{2}, 1+\tfrac{1}{3}, 1+\tfrac{1}{4}, \ldots\}$

Def A sequence that is either nondecreasing or nonincreasing is monotonic.

Def A sequence $\{a_n\}$ is bounded if for some real number N

$$|a_n| \le \underbrace{M}_{\text{const}} \quad \text{for any } n > N$$



Ex $\{a_n\} = \left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

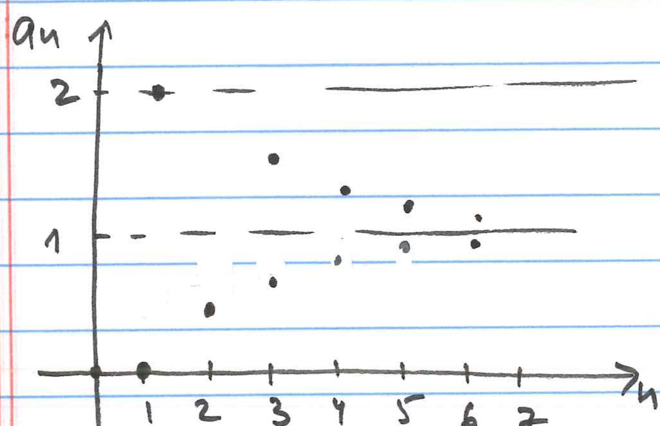
$$|a_n| < 1 \ : \ \text{bounded}, \ M = 1$$

Ex $\{a_n\} = \left\{1 + \frac{1}{n}\right\} = \left\{2, 1+\frac{1}{2}, 1+\frac{1}{3}, 1+\frac{1}{4}, \ldots\right\}$

$$|a_n| \le 2 \ : \ \text{bounded}, \ M = 2$$

Ex $\{a_n\} = \{\cos(n\pi)\}$

$$|a_n| = |\cos(n\pi)| = 1 \le 1 \ : \ \text{bounded}, \ M = 1$$



$\left\{1 + \frac{1}{n}\right\}$ nonincreasing bounded by 2

$\left\{1 - \frac{1}{n}\right\}$ nondecreasing bounded by 1

Thm 9.5 A bounded monotonic sequence converges.

Ex $\lim_{n\to\infty}(1-\frac{1}{n}) = \lim_{n\to\infty} 1 - \lim \frac{1}{n} = 1-0=1$: converges

nondecreasing
bounded

Ex $\lim(1+\frac{1}{n}) = 1$: converges

nonincreas.
bounded

Ex $\{a_n\} = \left\{\frac{n}{n+1}\right\}_{n=0}^{\infty} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$

nondecreasing, i.e. monotonic

$|a_n| < 1$: bounded, $M=1$

$\therefore$ by Thm 9.5, $\{a_n\}$ converges.

Indeed, $\lim_{n\to\infty} a_n = 1$.

Geometric Sequences

Geometric sequence is of the form $\{r^n\}$ or $\{ar^n\}$

Property: each term is obtained by

multiplying the previous term by a fixed number r, called <u>ratio</u>

$$\{r^n\}_{n=0}^{\infty} = \{1, r, r^2, r^3, \dots\}$$

$$\{ar^n\}_{n=0}^{\infty} = \{a, ar, ar^2, ar^3, \dots\}$$

Ex $\left\{\left(\frac{3}{4}\right)^n\right\}_{n=0}^{\infty}$ $\quad r = \frac{3}{4}, \quad a = 1$

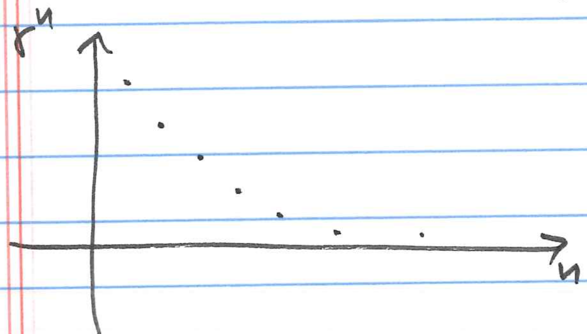
Ex $\{(-2)^n\}_{n=0}^{\infty}$ $\quad r = -2$

If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$

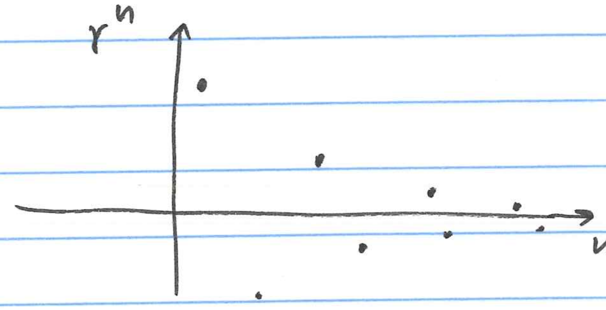
if $r = \pm 1 \Rightarrow \lim r^n$ does not exist since r^n oscillates between 1 and -1

if $|r| > 1 \Rightarrow \{r^n\}$ diverges

$r < 1$

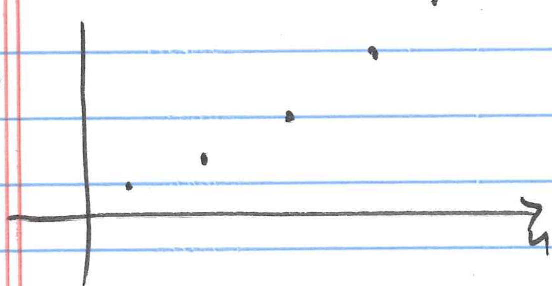


$r < 0, \ |r| < 1$

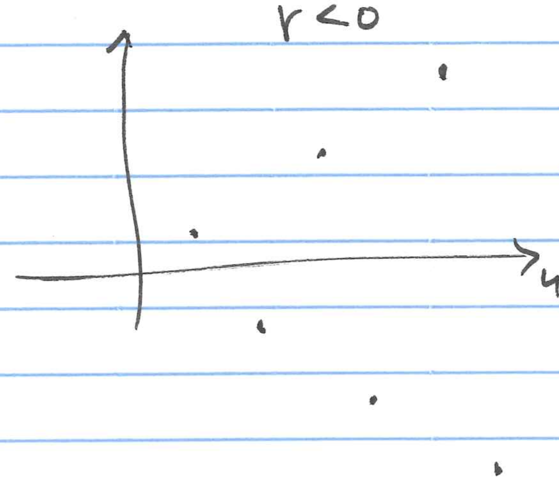


$|r|>1$

$r>0$



$r<0$



Thm 9.3 Geometric sequence $\{r^n\}$

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r|<1 \\ 1, & r=1 \\ \text{diverges, otherwise} & r\le -1 \text{ or } r>1 \end{cases}$$

Growth Rates of Sequences

As with functions, to compare growth rates of sequences, we evaluate $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$.

If $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=0 \Rightarrow \{b_n\}$ grows faster than $\{a_n\}$

If $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=0 \Rightarrow \{a_n\}$ grows faster than $\{b_n\}$

Ranking of growth rates:

$q, p, r > 0$ $\qquad \{\ln^q n\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+r}\} \ll$

$b > 1$ $\qquad \ll \{b^n\} \ll \{n^n\}$

<u>Note</u> $\{a_n\} \ll \{b_n\}$ means $\{b_n\}$ grows faster than $\{a_n\}$

Another sequence: <u>factorial sequence</u> $\{n!\}$ where

$$n! = n(n-1)(n-2)\dots 2\cdot 1$$

Consider

$$n^n = \underbrace{n \cdot n \dots \cdot n}_{n \text{ factors}}$$

$$n! = \underbrace{n(n-1)(n-2)\dots 2\cdot 1}_{n \text{ factors}}$$

$\Rightarrow \{n!\} \ll \{n^n\}$

$\Rightarrow \{b^n\} \ll \{n!\}$

$b > 1$
$$b^n = \underbrace{b \cdot b \dots \cdot b}_{n \text{ factors}}$$

$$\therefore \quad \{b^n\} \ll \{n!\} \ll \{n^n\}$$

Ex Compare growth rates

$\{\ln n^{10}\}$ and $\{0.00001 \cdot n\}$

$\{10 \ln n\} \ll \{10^{-5} \cdot n\}$

3/3/2014

Recall a formal def of $\lim\limits_{n\to\infty} a_n = L$

For $\underset{\text{any}}{\forall} \varepsilon > 0$, $\underset{\text{there exists}}{\exists} N = N(\varepsilon)$: $\underset{\text{any}}{\forall} n > N$ $\quad |a_n - L| < \varepsilon$

<u>Ex</u> $\quad a_n = \dfrac{n}{n+1} \qquad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

We need to show

$$|a_n - L| < \varepsilon \ ?$$

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} \overset{?}{<} \varepsilon$$

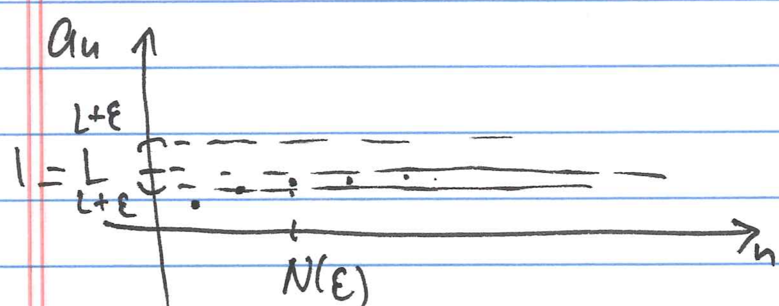
solve for n

$$\frac{1}{n+1} < \varepsilon$$

$$n+1 > \frac{1}{\varepsilon}$$

$$n > -1 + \frac{1}{\varepsilon} \Rightarrow N(\varepsilon) = \left\lfloor -1 + \frac{1}{\varepsilon} \right\rfloor + 1$$

integral part



e.g. $\varepsilon = \frac{1}{100} \Rightarrow N(\varepsilon) = \left\lfloor -1 + \frac{1}{1/100} \right\rfloor + 1 =$

$= 99 + 1 = 100$

Hence, for all $n > N(\varepsilon) = 100$, $|a_n - 1| < \varepsilon = \frac{1}{100}$

9.3 Infinite Series

Consider the geometric series $\sum_k r^k$ or $\sum_k ar^k$

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \dots + r^n + \dots$$

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^n + \dots =$$

$$= a(1 + r + r^2 + \dots + r^n + \dots)$$

Ex $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$: geometric series w/ $r = \frac{1}{2}$

Ex $\sum_{k=1}^{\infty} \frac{1}{k}$: is not a geometric series (This is a harmonic series)

Recall
$$\sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} S_n$$
partial sums

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^n + \dots$$

$$S_n = \underbrace{a + ar + ar^2 + \dots + ar^{n-1}}_{n \text{ terms}}$$

sum of first n terms

$$\begin{aligned} S_n &= a + ar + ar^2 + \dots + ar^{n-1} \\ - \quad rS_n &= \phantom{a + {}} ar + ar^2 + \dots + ar^{n-1} + ar^n \\ \hline (1-r)S_n &= a - ar^n \end{aligned}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$; n^{th} partial sum of geometric series

$$\therefore \sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} =$$

$$= \frac{a\left(1 - \lim_{n\to\infty} r^n\right)}{1-r}$$

Recall $\lim_{n\to\infty} r^n = \begin{cases} 0, & |r|<1 \\ 1, & r=1 \\ \text{does not exist otherwise} \end{cases}$

<u>Case 1</u>: $|r|<1 \Rightarrow \lim_{n\to\infty} r^n = 0$

$$\therefore \boxed{\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}}$$

<u>Case 2</u>: $|r|>1 \Rightarrow \lim_{n\to\infty} r^n$ is either ∞ or does not exist

$\therefore \sum_{k=0}^{\infty} ar^k$ diverges

Case 3: $|r|=1$

$r=1 \Rightarrow \sum_{k=0}^{\infty} a\cdot 1^k = a(1+1+\ldots)$ diverges

$r=-1 \Rightarrow \sum_{k=0}^{\infty} a(-1)^k$ diverges

$= a(1-1+1-1+\ldots)$

Summary:

$$\boxed{\text{Geometric series} \quad \sum_{k=0}^{\infty} ar^k = \begin{cases} \dfrac{a}{1-r}, & |r|<1 \\ \text{diverges}, & \text{otherwise} \end{cases}}$$

Ex Decimal expansions.

$0.\overline{456} = 0.456456456\ldots$

We can use geometric series and write this number as a fraction.

$$0.456456456\ldots = \overset{0.456}{\overset{\shortparallel}{0.456}} + \overbrace{0.000456}^{\overset{0.456\cdot 10^{-3}}{\shortparallel}} + \overbrace{0.000000456}^{\overset{0.456\cdot 10^{-6}}{\shortparallel}} + \ldots =$$

$$a + ar + ar^2 + \ldots = a(\underset{\overset{\shortparallel}{r^0}}{1} + r + r^2 + \ldots):$$ geometric series

Here $a = 0.456$, $r = 10^{-3}$

$$= 0.456\left(1 + 10^{-3} + (10^{-3})^2 + (10^{-3})^3 + \ldots\right) =$$

$$r = 10^{-3} < 1 \Rightarrow \sum_{k=0}^{\infty} 0.456\cdot(10^{-3})^k < \infty$$ converges

$$= \frac{a}{1-r} = \frac{0.456}{1-0.001} = \frac{0.456}{0.999} = \frac{456}{999}$$

#27 $\sum_{k=1}^{\infty} 2^{-3k}$: geom. series $\quad \sum ar^k$

$$2^{-3k} \overset{?}{\cancel{=}} 2\cdot 1^{-3k}$$

$r = \frac{1}{8} < 1$

$\Rightarrow$ converges $\qquad 2^{-3k} = (2^{-3})^k \Rightarrow r = 2^{-3} = \frac{1}{8}$

$$\sum_{k=1}^{\infty} (2^{-3})^k = \underbrace{\sum_{k=0}^{\infty} (2^{-3})^k}_{} - \underset{\text{term w/ } k=0}{1} = \overset{a=1}{\frac{8}{7}} - 1 = \boxed{\frac{1}{7}}$$

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = 8/7$$

Telescoping Series

Ex Evaluate $\sum_{k=1}^{\infty}\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)$

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)=\left(\frac{1}{3}-\cancel{\frac{1}{3^2}}\right)+$$

$$+\left(\cancel{\frac{1}{3^2}}-\cancel{\frac{1}{3^3}}\right)+\left(\cancel{\frac{1}{3^3}}-\cancel{\frac{1}{3^4}}\right)+\ldots+\left(\cancel{\frac{1}{3^n}}-\frac{1}{3^{n+1}}\right)$$

$$=\frac{1}{3}-\frac{1}{3^{n+1}}$$

$$\therefore \sum_{k=1}^{\infty}\left(\frac{1}{3^k}-\frac{1}{3^{k+1}}\right)=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(\frac{1}{3}-\overset{\nearrow 0}{\frac{1}{3^{n+1}}}\right)$$

$$\lim_{n\to\infty}\frac{1}{3^{n+1}}=\lim_{n\to\infty}\left(\frac{1}{3}\right)^n\cdot\frac{1}{3}=0$$

geometric seq
$r=\frac{1}{3}<1$

$$= \boxed{\frac{1}{3}}$$

Ex Evaluate $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$

$$\frac{1}{k(k+1)}=\frac{A}{k}^{(k+1}+\frac{B}{k+1}^{(k}=\frac{A(k+1)+Bk}{k(k+1)}$$

$$1 = A(k+1) + Bk$$

k^0: $\quad 1 = A$

k^1: $\quad 0 = A + B \Rightarrow B = -A = -1$

$$\therefore \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) +$$

$$+ \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \ldots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Then

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = \boxed{1}$$

#51 $\quad \sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty} \left(\ln(k+1) - \ln k\right)$:

telescopic series

$$\ln\frac{a}{b} = \ln a - \ln b$$

$$S_n = \sum_{k=1}^{n} \left(\ln(k+1) - \ln k \right) = \left(\cancel{\ln 2} - \ln 1 \right) +$$

$$+ \left(\ln 3 - \cancel{\ln 2} \right) + \left(\ln 4 - \ln 3 \right) + \dots + \left(\ln(n+1) - \ln n \right)$$

$$= -\cancelto{0}{\ln 1} + \ln(n+1) = \ln(n+1)$$

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k} \right) = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \ln(n+1) = \infty$$

$\therefore$ series diverges