3/4/2014

Exam #2 covers sections 8.3 – 9.3

Review: tomorrow, Wed., 4:30–5:20, TLC 223

#8.3.9. $\int_5^{10} \sqrt{100-x^2}\,dx =$

$5 \le x \le 10$

$x = 10\sin\theta$

$x=5 \Rightarrow 5 = 10\sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \boxed{\frac{\pi}{6}}$

$x=10 \Rightarrow 10 = 10\sin\theta$

$\Rightarrow \sin\theta = 1 \Rightarrow \theta = \boxed{\frac{\pi}{2}}$

$dx = 10\cos\theta\, d\theta$

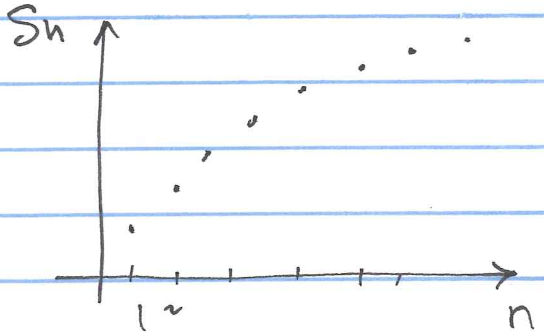
$$= \int_{\frac{\pi}{6}}^{\pi/2} \sqrt{100 - 100\sin^2\theta}\; 10\cos\theta\, d\theta = \ldots$$

9.4 The Divergence and Integral Test

Not always it is possible to find the sum of a series like we did for geometric or telescoping series. There are tests that allow one to answer whether a series converges or diverges. In some cases, a test may be inconclusive.

The Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$



$S_1 = 1$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$S_4 = \frac{25}{12}$

. . .

$S_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$

Q Does the harmonic series converge?

Thm 9.8 Properties of convergent series
We consider $\sum a_n$, $a_n > 0$.

1. Suppose $\sum a_n$ converges to A and let c be any real number. Then $\sum c a_n$ also converges and $\sum c a_n = cA$.

2. Suppose $\sum a_n$ converges to A and $\sum b_n$ converges to B. Then series $\sum (a_n \pm b_n)$ also converges and $\sum (a_n \pm b_n) = A \pm B$.

3. Whether a series converges does not depend on a finite number of terms added or removed from the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=M}^{\infty} a_n$ both converge or both diverge.

However, the value of a convergent series some does change if nonzero terms are added or removed.

Ex $$S=\sum_{k=1}^{\infty}\left[5\left(\frac{2}{3}\right)^{k}-\frac{2^{k-1}}{7^{k}}\right]$$

$\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^{k}$: geometric series $a=5,\ r=\frac{2}{3}$

Recall $\sum_{k=0}^{\infty} ar^{k}=\frac{a}{1-r}$ if $|r|<1$

$$\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^{k}=\sum_{k=0}^{\infty} 5\left(\frac{2}{3}\right)^{k}\underset{\substack{\uparrow\\ \text{term with } k=0}}{-5}=5\,\frac{1}{1-\frac{2}{3}}-5=$$

$$=5\left(\frac{1}{\frac{1}{3}}-1\right)=5(3-1)=\boxed{10}$$

or $$\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^{k}=5\cdot\frac{2}{3}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k-1}=\left|\begin{matrix} l=k-1\\ k=1\Rightarrow l=0\end{matrix}\right|=$$

$$=5\cdot\frac{2}{3}\underbrace{\sum_{l=0}^{\infty}\left(\frac{2}{3}\right)^{l}}_{\frac{1}{1-\frac{2}{3}}}=5\cdot\frac{2}{3}\cdot\frac{1}{1-\frac{2}{3}}=5\cdot\frac{2}{3}\cdot\frac{1}{\frac{1}{3}}=\boxed{10}$$

$a=1,\ r=\frac{2}{3}$

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{7^k} \underset{\uparrow}{=} \overbrace{\sum_{k=1}^{\infty} \left(\frac{2}{7}\right)^k}^{= \sum_{k=0}^{\infty} \left(\frac{2}{7}\right)^k - \left(\frac{2}{7}\right)^k \Big|_{k=0}} \cdot \underbrace{\frac{1}{2}}_{2^{-1}} \textcircled{=}$$

$2^{k-1} = 2^k \cdot 2^{-1}$

geometric series w/ $a = \frac{1}{2}$, $r = \frac{2}{7}$

converges since $\frac{2}{7} < 1$

$$\textcircled{=} \frac{1}{2}\left(\sum_{k=0}^{\infty} \left(\frac{2}{7}\right)^k - 1\right) = \frac{1}{2}\left(\frac{1}{1-\frac{2}{7}} - 1\right) =$$

$$= \frac{1}{2}\left(\frac{1}{\frac{5}{7}} - 1\right) = \frac{1}{2}\left(\frac{7}{5} - 1\right) = \frac{1}{2}\left(\frac{7-5}{5}\right) = \boxed{\frac{1}{5}}$$

Hence,

$$S = \sum_{k=1}^{\infty} \left[5\left(\frac{2}{3}\right)^k - \frac{2^{k-1}}{7^k}\right] =$$

$$= \underbrace{\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^k}_{\text{converges}} - \underbrace{\sum_{k=1}^{\infty} \frac{2^{k-1}}{7^k}}_{\text{converges}} = 10 - \frac{1}{5} = \boxed{\frac{49}{5}}$$

Thm 9.9. Divergence Test or Necessary Condition of Convergence

If $\sum a_n$ converges, then $\lim_{k\to\infty} a_n = 0$. Equivalently, if $\lim_{k\to\infty} a_n \neq 0$, then series diverges.

Proof Assume that $\sum a_k$ converges. Then the sequence $\{S_k\}$ of partial sums has a limit: $\lim\limits_{k\to\infty} S_k = S$.

$$\sum a_k = \lim_{k\to\infty} S_k = S$$

Note $a_k = S_k - S_{k-1}$ since

$$S_k = \underbrace{a_1 + a_2 + \ldots + a_{k-1}}_{S_{k-1}} + a_k$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim S_{k-1} =$$

$$= S - S = 0$$

Hence, if $\sum a_k < \infty \Rightarrow \lim\limits_{k\to\infty} a_k = 0$

Property: if (p) $\overset{\text{then}}{\Longrightarrow}$ (q)

Contrapositive: if (not q) $\overset{\text{then}}{\Rightarrow}$ (not p)

In our case: if $\sum a_k < \infty \overset{\text{then}}{\Rightarrow} \lim\limits_{k\to\infty} a_k = 0$

Contrapositive: if $\lim\limits_{k\to\infty} a_k \neq 0 \overset{\text{then}}{\Rightarrow} \sum a_k$ diverges

Note: if $\lim\limits_{k\to\infty} a_k = 0$, it does not imply

$$\sum a_k < \infty$$

$$\text{if } \sum a_k < \infty \Rightarrow \lim_{k\to\infty} a_k = 0$$

$$\not\Leftarrow$$

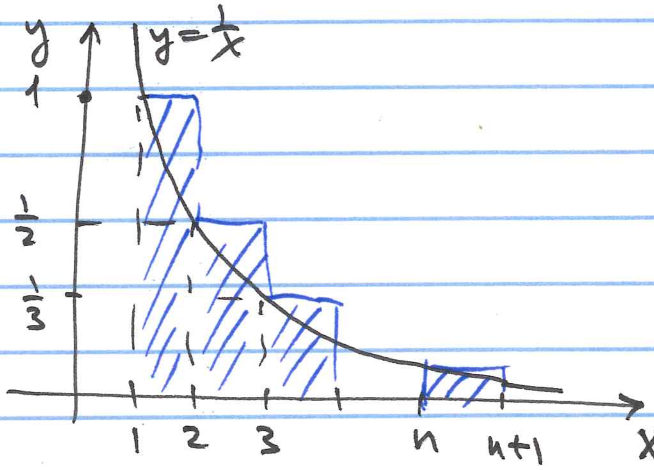
Ex $\sum_{k=1}^{\infty} \frac{1}{k}$ $\quad a_k = \frac{1}{k} \to 0$ as $k \to \infty$

but $\sum \frac{1}{k}$ diverges (will show)

Consider n^{th} partial sum S_n of harm. series

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}_{\text{left Riemann sum of } y=\frac{1}{x} \text{ on } [1, n+1]}$$

$\int_1^{n+1} \frac{1}{x}\,dx$: area under $y = \frac{1}{x}$, $x \in [1, n+1]$



From graph,

$$S_n > \int_1^{n+1} \frac{dx}{x}$$

$$\int_1^{n+1} \frac{dx}{x} = \ln x \Big|_1^{n+1} = \ln(n+1) - \cancel{\ln 1}^{\,0} = \ln(n+1) \nearrow$$

as $n \nearrow$

$$\therefore \lim_{n\to\infty} S_n = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

Thm 9.10 Harmonic series

The harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

diverges even though $a_k = \frac{1}{k} \to 0$ as $k \to \infty$.

Ex $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$

$a_k = \frac{k}{k+1}$ $\qquad \lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \frac{k}{k+1} = 1 \neq 0$

$\Rightarrow \sum a_k$ diverges by Divergence Test

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ $\qquad a_k = \frac{1}{\sqrt{k}} \to 0$: Divergence Test is inconclusive

We actualy can use p-test to show that $\sum \frac{1}{\sqrt{k}}$ diverge

$p = \frac{1}{2} < 1$

$\sum a_k$, $a_k = f(k)$

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Thm 9.11 **Integral Test**

Suppose f is a continuous, positive, decreasing function for $x \geq 1$, and let $a_k = f(k)$, $k = 1, 2, \ldots$ Then

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge. In the case of convergence, the value of integral is NOT in general, equal to value of the series.

Ex $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{k}{k^2+1}}_{a_k}$ $\qquad a_k = f(k)$

$$f(x) = \frac{x}{x^2+1} \quad \text{is continuous}$$

for $x \geq 1$, $f(x) > 0$

We can show

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \Rightarrow f$ is decreasing

or

we can show that $f' < 0 \Rightarrow f$ is decreasing

$$f'(x) = \frac{1(x^2+1) - 2x \cdot x}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0$$

$\Rightarrow f(x)$ is decreasing $\qquad x > 1$

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x}{x^2+1}\,dx = \left| \begin{matrix} u = x^2+1 \\ du = 2x\,dx \\ x=1 \Rightarrow u=2 \\ x=\infty \Rightarrow u=\infty \end{matrix} \right| =$$

$$= \int_2^\infty \frac{\frac{1}{2}\,du}{u} = \frac{1}{2} \ln u \Big|_2^\infty \ \text{⊜}$$

$$2 \le u < \infty \qquad \shortparallel\ \frac{1}{2} \lim_{b\to\infty} \ln b \Big|_2^b$$

$$\text{⊜}\ \frac{1}{2}\left(\lim_{u\to\infty} \ln u - \ln 2 \right) = \infty$$

$\Rightarrow \int_1^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{k=1}^\infty \frac{k}{k^2+1}$ diverges by integral Test.

The p-series : $\sum_{k=1}^\infty \frac{1}{k^p}$

When $p=1$ we get $\sum_{k=1}^\infty \frac{1}{k}$, harmonic series, that diverges.

Let $p>0$. We will use Integral Test to determine convergence of $\sum_{k=1}^\infty \underbrace{\frac{1}{k^p}}_{a_k = f(k)}$.

$f(x) = \frac{1}{x^p}$

$\int_1^\infty \frac{dx}{x^p}$ converges when $p > 1$
diverges when $p \leq 1$

Hence, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges when $p > 1$ and diverges when $p \leq 1$ by Integral Test.

Ex $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges as p-series w/ $p = 4 > 1$

Ex $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges as p-series w/ $p = \frac{1}{2} < 1$

Thm 9.12 Convergence of p-series

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$

estimating the value (sum) of an infinite series

For a convergent series $\sum a_k$, we can approximate its sum if we use first n terms of the series.

$$\underset{\substack{\text{exact} \\ \text{sum}}}{\sum_{k=1}^{\infty} a_k} = \underbrace{a_1 + a_2 + \dots + a_n}_{\substack{S_n \\ \| \\ \sum_{k=1}^{n} a_k \\ \text{approximation}}} + \underbrace{a_{n+1} + a_{n+2} + \dots}_{\substack{\text{remainder} \\ R_n \\ \text{or error}}}$$

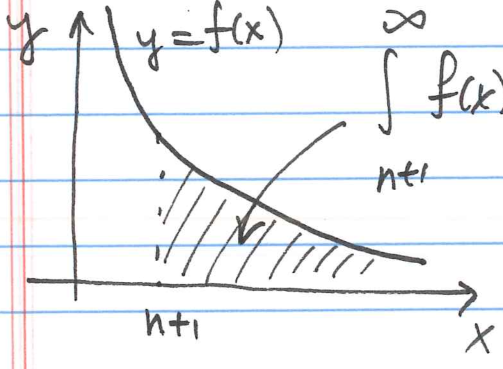
$$R_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \underset{\substack{\| \\ f(n+1)}}{a_{n+1}} + \underset{\substack{\| \\ f(n+2)}}{a_{n+2}} + \dots$$

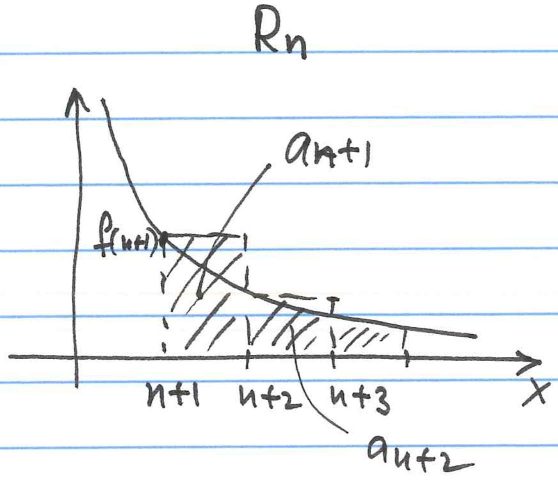
remainder

or tail of the series

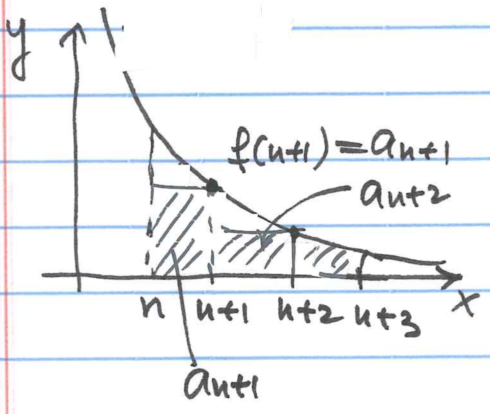
Let f be continuous, $f>0$, decreasing,

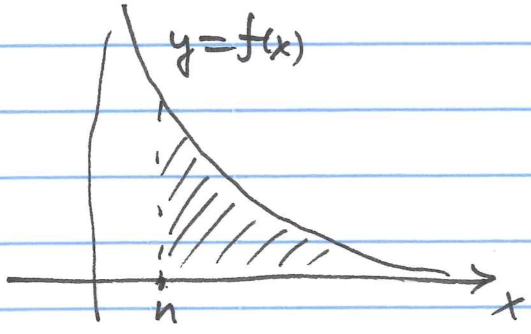
$f(k) = a_k$





From graphs: $\int_{n+1}^{\infty} f(x)\,dx \leq R_n$





$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots$

From graph, $R_n \leq \int_{n}^{\infty} f(x)\,dx$

Hence

$$\underbrace{\int_{n+1}^{\infty} f(x)\,dx}_{\substack{\text{can be}\\\text{computed}\\\text{since } f \text{ is}\\\text{known}}} \leq R_n \leq \underbrace{\int_{n}^{\infty} f(x)\,dx}_{\substack{\text{can be}\\\text{computed}}} \tag{1}$$

Then we can estimate R_n.

<u>**Note**</u> $S = \sum_{k=1}^{\infty} a_k = \underbrace{\sum_{k=1}^{n} a_k}_{S_n} + \underbrace{\sum_{k=n+1}^{\infty} a_k}_{R_n} = S_n + R_n$

Add S_n to all sides of (1).

(2) $$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx}_{L_n} \leq \underbrace{S_n + R_n}_{\overset{\shortparallel}{S = \sum_{k=1}^{\infty} a_k}} \leq \underbrace{S_n + \int_{n}^{\infty} f(x)\,dx}_{U_n}$$

S_n can be computed as the sum of first n terms of $\sum_{k=1}^{\infty} a_k$.

If we compute $\int_{n+1}^{\infty} f(x)\,dx$ and $\int_{n}^{\infty} f(x)\,dx$,

then
$$L_n = S_n + \int_{n+1}^{\infty} f(x)\,dx$$

and $$U_n = S_n + \int_{n}^{\infty} f(x)\,dx$$

are lower and upper bounds of $\sum_{k=1}^{\infty} a_k$

Inequalities (2) can be used to estimate the sum of $\sum_{k=1}^{\infty} a_k$.

Thm 9.13 Estimating Series with Positive Terms

Let f be a continuous, positive, decreasing function for $x \geq 1$ and let $a_k = f(k)$, $k = 1, 2, \ldots$ Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series

and $S_n = \sum_{k=1}^{n} a_k$ be the sum of its first

n terms. The remainder $R_n = S - S_n$ satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Furthermore, the exact value (sum) of the series is bounded:

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq S_n + \int_{n}^{\infty} f(x)\,dx$$