3/10/2014

Ex Estimating p-series: $\sum_{k=1}^{\infty} \frac{1}{k^p}$

Q How many terms of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to keep to get an approximation within 10^{-3}?

$$f(k) = \frac{1}{k^2}$$

From Thm 9.13, the remainder R_n satisfies

$$R_n \leq \int_n^{\infty} f(x)\,dx = \int_n^{\infty} \frac{1}{x^2}\,dx =$$

$$= -\frac{1}{x}\Big|_n^{\infty} = -\cancel{\lim_{x\to\infty} \frac{1}{x}}^{\,0} + \frac{1}{n} = \frac{1}{n}$$

$$\therefore \quad R_n \leq \underbrace{\frac{1}{n}}_{\text{upper bound of } R_n} \leq \underbrace{10^{-3}}_{= \frac{1}{10^3}}$$

Recall, remainder R_n is essentially the error between exact value S $S = \sum_{k=1}^{\infty} a_k$ and its approximation $S_n = \sum_{k=1}^{n} a_k$

$\frac{1}{n} \leq 10^{-3}$: solve for n

$n \geq 10^3 = 1000$

$\therefore$ If we keep at least $n = 1000$ terms, i.e. approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ with $\sum_{k=1}^{n} \frac{1}{k^2}$, then the error will not exceed 10^{-3}.

$2 < 3$

$\frac{1}{2} > \frac{1}{3}$

Ex Approximate $\sum_{k=1}^{\infty} \frac{1}{k^2}$ using 50 terms.

From Thm 9.13

$$\underbrace{S_n + \int_{n+1}^{\infty} f(x)\,dx}_{\substack{L_n \\ \text{lower bound}}} \le \underbrace{\sum_{k=1}^{\infty} a_k}_{\substack{S \\ \text{exact} \\ \text{value / sum}}} \le \underbrace{S_n + \int_{n}^{\infty} f(x)\,dx}_{\substack{U_n \\ \text{upper bound}}}$$

$n=50$

$$L_n = S_n + \int_{n+1}^{\infty} f(x)\,dx = S_n + \int_{n+1}^{\infty} \frac{dx}{x^2} = S_n + \frac{1}{n+1}$$

$$U_n = S_n + \int_{n}^{\infty} f(x)\,dx = S_n + \int_{n}^{\infty} \frac{dx}{x^2} = S_n + \frac{1}{n}$$

$$\therefore \quad S_n + \frac{1}{n+1} \le S \le S_n + \frac{1}{n}$$

$n=50$

$$S_{50} + \frac{1}{50+1} \le S \le S_{50} + \frac{1}{50}$$

$$S_{50} = \sum_{k=1}^{50} \frac{1}{k^2} \approx 1.625133$$

$$\Rightarrow \quad 1.625133 + \frac{1}{51} \le S \le 1.625133 + \frac{1}{50}$$

$$1.644741 \le S \le 1.645133$$

Then S can be approximated as average of lower and upper bounds, i.e.

$$S \approx \frac{1.644741 + 1.645133}{2} \approx 1.644937$$

closer to S than S_{50}

9.5 The Ratio, Root and Comparison Tests

These are convergence tests.

Thm 9.14 The Ratio Test

Let $\sum a_k$ be an infinite series, $a_k > 0$ and let $r = \lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k}$.

1. If $0 \le r < 1$, the series converges.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. If $r = 1$, the test is inconclusive.

Idea of proof

For $k \gg 1$ (k very large), $r \approx \frac{a_{k+1}}{a_k}$

$$\Rightarrow a_{k+1} \approx r a_k$$

Consider the tail of the series:

$$a_k + a_{k+1} + a_{k+2} + \dots \approx a_k + r a_k + r^2 a_k + \dots$$

$$= a_k(\underbrace{1+r+r^2+\dots}_{\text{geometric series}})$$

geometric series with $r<1$: convergent

if $r>1$: divergent $\blacksquare$

Ex $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{10^k}{k!}}_{a_k}$ $\qquad a_k = \dfrac{10^k}{k!}$, $\quad a_{k+1} = \dfrac{10^{k+1}}{(k+1)!}$

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k} =$$

$$= \lim_{k\to\infty} \frac{\cancel{10^k}\cdot 10^1 \cdot \cancel{k!}}{\cancel{k!}(k+1)\cancel{10^k}} = \lim_{k\to\infty} \frac{10}{k+1} = 0 < 1$$

$$(k+1)! = \underbrace{1\cdot 2\cdot 3 \dots (k-1)(k)}_{k!}(k+1) = (k+1)\,k!$$

Since $r=0<1$, series $\displaystyle\sum_{k=1}^{\infty}\frac{10^k}{k!}$ converges by Ratio test

Ex $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{k^k}{k!}}_{a_k}$

$$r = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{\underbrace{(k+1)!}_{=k!(k+1)}} \cdot \frac{k!}{k^k} =$$

$$= \lim_{k\to\infty} \frac{(k+1)^{k+1}\cdot \cancel{k!}}{\cancel{k!}(k+1)\cdot k^k} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)\,k^k} =$$

$$= \lim_{k\to\infty} \frac{(k+1)^k \cancel{(k+1)}}{\cancel{(k+1)}\, k^k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k =$$

$$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = \lim_{k\to\infty} e^{\ln\left(1+\frac{1}{k}\right)^k} =$$

$$= e^{\lim\limits_{k\to\infty} k \ln\left(1+\frac{1}{k}\right)} = e^L$$

$$L = \lim_{k\to\infty} k \ln\left(1+\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{\frac{1}{k}} = \left| t = \frac{1}{k} \right|$$

$$= \lim_{t\to 0} \frac{\ln(1+t)}{t} = \frac{0}{0} \overset{\text{l'Hop.}}{\underset{\text{rule}}{=}} \lim_{t\to 0} \frac{\frac{1}{1+t}}{1} = 1$$

$$\therefore \quad r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = e^L = e^1 = e > 1$$

since $r > 1$, series $\sum\limits_{k=1}^{\infty} \frac{k^k}{k!}$ diverges by Ratio Test

Note We can use n^{th} term test or Divergence test as well. Indeed,

$$a_k = \frac{k^k}{k!}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^k}{k!} = \lim_{k\to\infty} \frac{\overbrace{k \cdot k \cdots k}^{k \text{ times}}}{\underbrace{1 \cdot 2 \cdots (k-1)k}_{k \text{ times}}} = \infty$$

since $\lim\limits_{k\to\infty} a_k \neq 0 \Rightarrow \sum\limits_{k=1}^{\infty} \frac{k^k}{k!}$ diverges by Divergence Thm

Thm 9.5 The Root Test

Let $\sum a_k$ be an infinite series, $a_k \geq 0$ and let $\rho = \lim\limits_{k\to\infty} \sqrt[k]{a_k}$

1. If $0 \leq \rho < 1$, the series converges
2. If $\rho > 1$, including $\rho = \infty$, the series diverges
3. If $\rho = 1$, the test is inconclusive.

Idea of proof : for $k \gg 1$, $\rho \approx \sqrt[k]{a_k}$

$$\Rightarrow a_k \approx \rho^k$$

Series tail:

$$a_k + a_{k+1} + a_{k+2} + \ldots \approx \rho^k + \rho^{k+1} + \rho^{k+2} + \ldots =$$

$$= \rho^k \left[1 + \rho^2 + \rho^3 + \ldots \right]$$

geometric series with $r = \rho$: converges if $0 \leq \rho < 1$ and diverges if $\rho > 1$ $\blacksquare$

Ex $\sum\limits_{k=1}^{\infty} \underbrace{\left(\dfrac{4k^2-3}{7k^2+6} \right)^k}_{a_k}$

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{4k^2-3}{7k^2+6} = \frac{4}{7} < 1 \Rightarrow$$

$\sqrt[k]{a_k} = (a_k)^{\frac{1}{k}}$

$\Rightarrow \sum\limits_{k=1}^{\infty} \left(\dfrac{4k^2-3}{7k^2+6} \right)^k$ converges by Root test

Ex $\sum_{k=1}^{\infty} \underbrace{\frac{2^k}{k^{10}}}_{a_k}$

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \left(\frac{2^k}{k^{10}}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{2^{k\cdot\frac{1}{k}}}{k^{\frac{1}{k}\cdot 10}} =$$

$$= \lim_{k\to\infty} \frac{2}{\left(k^{\frac{1}{k}}\right)^{10}} \ominus$$

$$\lim_{k\to\infty} k^{\frac{1}{k}} = \lim_{k\to\infty} e^{\ln k^{\frac{1}{k}}} = e^{\lim_{k\to\infty} \frac{1}{k} \ln k} =$$

$$= e^0 = 1$$

$\ominus \lim_{k\to\infty} \frac{2}{1^{10}} = 2 > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{2^k}{k^{10}}$ diverges by Root test

3/11/2014

Thm 9.16 Comparison Test

Let $\sum a_k$ and $\sum b_k$ with $a_k, b_k > 0$.

1. If $0 < a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.
2. If $0 < b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Proof Assume $\sum b_k$ converges, i.e. $\sum b_k = B$ finite

$$S_n = \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} b_k < \sum_{k=1}^{\infty} b_k = B$$

$\Rightarrow \{S_n\}$ is a bounded sequence

Since $a_k > 0 \Rightarrow \{S_n\}$ is non-decreasing

} by Bounded Monotonic sequence Thm 9.5 $\{S_n\}$ converges

Hence, $\sum a_k$ converges.

Ex $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k^2+10}$

$$\underbrace{\frac{1}{k^2+10}}_{a_k} < \underbrace{\frac{1}{k^2}}_{b_k} \qquad k \geq 1$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges as p-series w/ $p = 2 > 1$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges by Comparison Test

$$\sum \frac{1}{k^2+10} < \sum \frac{1}{k^2} < \infty \quad \Rightarrow \quad \sum \frac{1}{k^2+10} < \infty$$

$\sum \frac{1}{k^2}$ converges, $\sum \frac{1}{k^2+10}$ converges

Ex $\displaystyle\sum_{k=4}^{\infty} a_k = \sum_{k=4}^{\infty} \frac{1}{\sqrt{k-3}}$ $\qquad \frac{1}{\sqrt{k-3}} \approx \frac{1}{\sqrt{k}}$

$\sum \frac{1}{\sqrt{k}}$ diverges as p-series w/ $p = \frac{1}{2} < 1$

$$\underbrace{\frac{1}{\sqrt{k-3}}}_{a_k} > \underbrace{\frac{1}{\sqrt{k}}}_{b_k}$$

$$\sum \frac{1}{\sqrt{k-3}} > \sum \frac{1}{\sqrt{k}} \Rightarrow \sum \frac{1}{\sqrt{k-3}}$$ diverges by Comparison Test

$\sum \frac{1}{\sqrt{k}}$ diverges

Thm 9.17 Limit Comparison Test

Suppose $\sum a_k$, $\sum b_k$ with $a_k, b_k > 0$ and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

1. If $0 < L < \infty$, L is finite, $L \neq 0$

   Then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

2. If $L = 0$ and $\sum b_k$ converges $\Rightarrow \sum a_k$ converges
3. If $L = \infty$ and $\sum b_k$ diverges $\Rightarrow \sum a_k$ diverges

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{k^4-2k^2+3}{2k^6-k+5}$

$$a_k = \frac{k^4-2k^2+3}{2k^6-k+5} \approx \frac{k^4}{2k^6} = \frac{1}{2k^2} = b_k$$

$$\sum \frac{1}{2k^2} = \frac{1}{2}\sum\frac{1}{k^2} < \infty \text{ as p-series w/ } p=2>1$$

$$L = \lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{k^4-2k^2+3}{2k^6-k+5}\cdot\frac{1}{\frac{1}{2k^2}} =$$

$$= \lim_{k\to\infty}\frac{k^4-2k^2+3}{2k^6-k+5}\cdot 2k^2 = \underset{\text{finite}}{1 \neq 0}$$

$$\therefore \sum \frac{k^4-2k^2+3}{2k^6-k+5} < \infty \text{ by Limit Comparison Test}$$

Ex $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{k^2}$

We can compare $\sum \frac{\ln k}{k^2}$ with $\sum\frac{1}{k^2}$ or $\sum\frac{1}{k}$

because $\underbrace{\sum\frac{\ln k}{k^2}}_{a_k}$ is "between $\sum\frac{1}{k^2}$ and $\sum\frac{1}{k}$

converges as p-series w/ $p=2>1$

diverges as harmonic series

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{\ln k}{k^2} = 0$$

I) Compare w/ $\sum \underbrace{\frac{1}{k^2}}_{b_k}$

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k}{k^2} \cdot \frac{1}{\frac{1}{k^2}} = \lim_{k\to\infty} \ln k = \infty$$

Since $\sum b_k = \sum \frac{1}{k^2}$ converges and $L = \infty$, the Limit Comparison Test is inconclusive.

3/13/2014

Ex Use Limit Comparison Test to determine convergence or Divergence of $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ (Cont'd)

II) Compare $\sum_{k=1}^{\infty} \underbrace{\frac{\ln k}{k^2}}_{a_k}$ with $\sum_{k=1}^{\infty} \underbrace{\frac{1}{k}}_{b_k}$

harmonic series diverges

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k}{k^2} \cdot \frac{1}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\ln k}{k^2} \cdot k$$

$$= \lim_{k\to\infty} \frac{\ln k}{k} = 0$$ as k grows faster than $\ln k$ as $k \to \infty$ (or use l'Hopital's Rule)

hence, $L = \lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum b_k = \sum \frac{1}{k}$ diverges

$\Rightarrow \sum a_k$ may converge or diverge

so the Limit Comparison Test is again inconclusive.

III) Compare with $\sum_{k=1}^{\infty} \overbrace{\frac{1}{k^{3/2}}}^{b_k}$: p-series w/ $p = \frac{3}{2}$

converges since $\frac{3}{2} > 1$

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\ln k}{k^2} \cdot \frac{1}{\frac{1}{k^{3/2}}} = \lim_{k\to\infty} \frac{\ln k \cdot k^{3/2}}{k^2}$$

$$= \lim_{k\to\infty} \frac{\ln k}{k^{2-3/2}} = \lim_{k\to\infty} \frac{\ln k}{k^{1/2}} = 0$$

$2 - \frac{3}{2}$

hence, $L = \lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} b_k < \infty$

then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges by

Limit Comparison Test.

9.6 Alternating Series

Before we consider series $\sum a_k$ with $a_k > 0$.

Now we will allow terms of a series to have alternating sign.

Ex $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$:

alternating harmonic series

Consider partial sums S_n:

$S_1 = 1$

$S_2 = 1 - \frac{1}{2} = \frac{1}{2}$

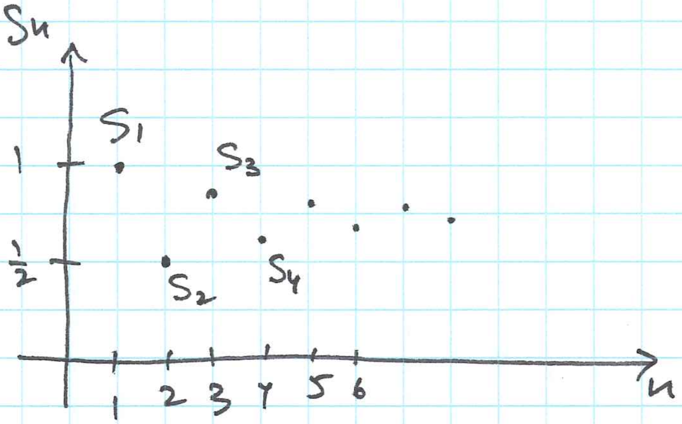
$S_3 = \underbrace{1 - \frac{1}{2}}_{S_2} + \frac{1}{3} = \frac{5}{6} = S_2 + \frac{1}{3}$

$S_4 = \underbrace{1 - \frac{1}{2} + \frac{1}{3}}_{S_3} - \frac{1}{4} = \frac{7}{12} = S_3 - \frac{1}{4}$

$S_5 = S_4 + \frac{1}{5}$

etc.

Q Does the alternating harmonic series converge?



From the graph, it seems that sequence $\{S_n\}$ of partial sums converges and is bounded.

Thm 9.18 The Alternating Series Test

The alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges provided:

1. the terms of the series are nonincreasing in magnitude: $0 < a_{k+1} \leq a_k$ for all $k > N$ ← some index
2. $\lim_{k\to\infty} a_k = 0$

Note For $\sum a_k$, $a_k > 0$, $\lim_{k\to\infty} a_k = 0 \not\Rightarrow \sum a_k$ converges

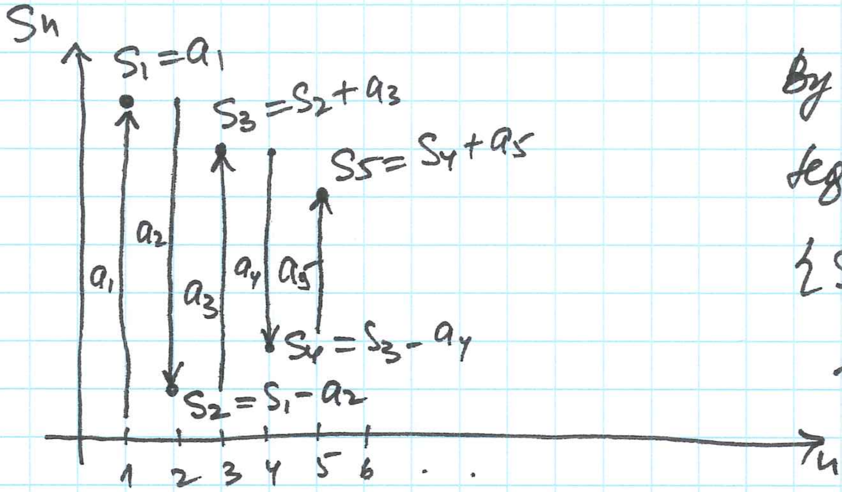
For $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, $a_k > 0$, $\left.\begin{array}{l} \lim a_k = 0 \\ a_{k+1} \leq a_k \end{array}\right\} \Rightarrow \sum a_k$ converges

Proof Consider $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = \underbrace{\underbrace{\overbrace{a_1 - a_2}^{\geq 0}}_{S_2} + \overbrace{a_3 - a_4}^{\geq 0}}_{S_4} + \overbrace{a_5 - a_6}^{\geq 0} + \dots$

Since $a_{k+1} \leq a_k \Rightarrow$ the even terms $\{S_{2k}\} =$

$= \{S_2, S_4, S_6, \dots\}$ form a nondecreasing sequence

and it is bounded from above by S_1



By Bounded Monotonic sequence Thm, the sequence $\{S_{2k}\}$ converges, say, to limit L.

Similarly, the odd terms $\{S_{2k+1}\} = \{S_1, S_3, S_5 \dots,\}$ form a nonincreasing sequence bounded from below by S_2. By Bounded Monotonic sequence Thm, sequence $\{S_{2k+1}\}$ also converges to limit, say, L'. We do not know if $L = L'$. We will show that $L = L'$.

<u>Notice:</u> $\quad S_{2k+1} = S_{2k} + a_{2k+1}$

or $\quad \boxed{S_{2k} = S_{2k-1} - a_{2k}}$

We know $\quad \lim\limits_{k\to\infty} S_{2k} = L, \quad \lim\limits_{k\to\infty} S_{2k-1} = L'$

$$\lim_{k\to\infty} a_k = 0 \Rightarrow \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+1} = 0$$

$$\therefore \underbrace{\lim_{k\to\infty} S_{2k}}_{= L} = \underbrace{\lim_{k\to\infty} S_{2k-1}}_{L'} - \underbrace{\lim_{k\to\infty} a_{2k}}_{= 0} \Rightarrow L = L'$$

Therefore, the sequence $\{S_n\}$ has a unique limit L. Hence, series $\sum_{k=1}^{\infty}(-1)^{k+1}$ also converges by def. ■

Ex $\sum_{k=1}^{\infty}(-1)^{k+1}\underbrace{\frac{1}{k}}_{a_k}$: alternating harmonic series

$\left.\begin{array}{l}\frac{1}{k+1}<\frac{1}{k} \Rightarrow a_{k+1}<a_k \\ \lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{1}{k} = 0\end{array}\right\}$ $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges by Alternating Series Test

Ex $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$ $\qquad a_k=\frac{1}{k^2}$

$\left.\begin{array}{l}\frac{1}{(k+1)^2}<\frac{1}{k^2} \Rightarrow a_{k+1}<a_k \\ \lim_{k\to\infty}\frac{1}{k^2}=0\end{array}\right\} \Rightarrow \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$ converges by altern. series Test

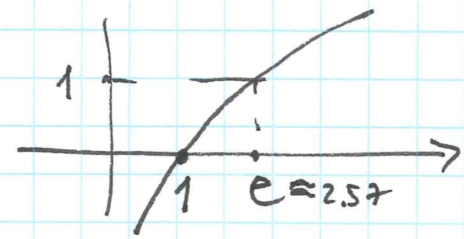
Ex $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\ldots=\sum_{k=1}^{\infty}\underbrace{\frac{k+1}{k}}_{a_k}\cdot(-1)^{k+1}$

$\lim_{k\to\infty} a_k = 1 \neq 0 \Rightarrow \sum\frac{k+1}{k}(-1)^{k+1}$ diverges by Divergence or n^{th} term Thm

Ex $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{k}$

$$a_k = \frac{\ln k}{k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln k}{k} = 0$$



$$f(x) = \frac{\ln x}{x}$$

$$f' = \frac{1}{x^2}\left(\frac{1}{x}\cdot x - 1\cdot \ln x\right) = \frac{1}{x^2}(1-\ln x)$$

$$f' < 0 \quad \text{if} \quad x > e$$

$$\Rightarrow \quad f' < 0 \quad \text{if} \quad x \geq 3$$

$f' < 0 \Rightarrow f$ decreases $\Rightarrow$ $a_{k+1} \leq a_k$ for $k \geq 3$

Hence, $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k \ln k}{k}$ converges by Alt. Series Test.

3/14/2014

Remainders in Alternating Series

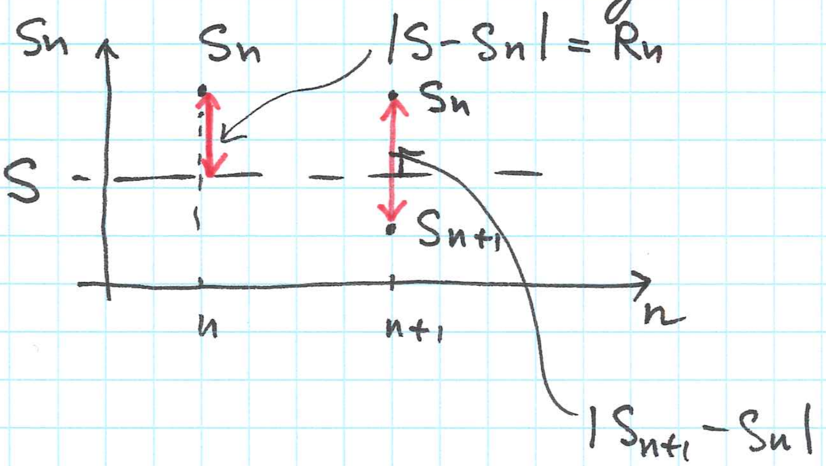
$$S = \sum_{k=1}^{\infty} a_k, \quad a_k > 0$$

$$S = \underbrace{a_1 + a_2 + \ldots + a_n}_{S_n} + \underbrace{a_{n+1} + \ldots}_{R_n}$$

$R_n = S - S_n \geqslant 0$ $\qquad |S - S_n| = S - S_n$, if $a_k > 0$

For alternating series, it is convenient to define

$R_n = |S - S_n|$: absolute error in approximation of an infinite series with sum S by the sum S_n of first n terms



<u>Note</u>: sum S is trapped between successive values S_n

$$S = \underbrace{\underbrace{a_1 - a_2 + a_3 - a_4 + \ldots + a_n}_{S_n} - a_{n+1}}_{S_{n+1}} + a_{n+2} - a_{n+3} + \ldots$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k, \quad a_k > 0$$

$$S_{n+1} - S_n = \pm a_{n+1} \Rightarrow |S_{n+1} - S_n| = a_{n+1}$$

From graph, we see that

$$R_n = |S - S_n| \le |S_{n+1} - S_n| = a_{n+1}$$

$$\therefore \quad \boxed{R_n \le a_{n+1}}$$

<u>Thm 9.20</u> <u>Remainder in alternating Series</u>

Let $R_n = |S - S_n|$ be the remainder in approximating the value of a convergent altern. series $\sum (-1)^{k+1} a_k$ by S_n. Then

$$R_n \le a_{n+1}$$

i.e. the remainder is $\le$ magnitude of the first neglected term.

<u>Ex</u> $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$: Taylor series of $\ln(1+x)$ (more - later)

$x = 1 \Rightarrow$

$$\ln 2 = \ln(1+1) =$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

<u>Q</u> How many terms are required to approximate $\ln 2$ with remainder $< 10^{-6}$?

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = \underbrace{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n+1}\cdot\frac{1}{n}}_{S_n} + (-1)^{n+2}\cdot\underbrace{\frac{1}{n+1}}_{a_{n+1}}+\ldots$$

$$R_n \le a_{n+1} = \frac{1}{n+1} < 10^{-6}$$

$$n+1 > \frac{1}{10^{-6}} = 10^{6}$$

$$\Rightarrow \quad n > 10^{6}-1$$

If we keep n at least 10^6, then remainder R_n will be $< 10^{-6}$.

$$\int\frac{dx}{\sec x+1}$$

Absolute and Conditional Convergence

We will consider a general $\sum a_n$ with terms a_n that can be positive or alternating sign etc.

Recall

<u>Ex</u> $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$: alternating harmonic series, converges

but $\displaystyle\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k}\right| = \sum_{k=1}^{\infty}\frac{1}{k}$: harmonic series diverges

<u>Ex</u> $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^2}$: converges by Alt. Series Test

and $\displaystyle\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k^2}\right| = \sum_{k=1}^{\infty}\frac{1}{k^2}$: converges as p-series w/ $p=2>1$

Def Assume that $\sum a_k$ converges. The series $\sum a_k$ converges absolutely if $\sum |a_k|$ converges. Otherwise, $\sum a_k$ converges conditionally.

Ex $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$: converges

$\sum \left| (-1)^{k+1} \frac{1}{k^2} \right| = \sum \frac{1}{k^2}$ converges

$\therefore \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$ converges absolutely

Ex $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$: converges

but $\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges

$\therefore \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ converges conditionally

Thm 9.21 Absolute convergence implies convergence.

If $\sum |a_k|$ converges, then $\sum a_k$ converges, i.e. abs. convergence implies convergence.

If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

$\sum |a_k|$ converges $\Longrightarrow$ $\sum a_k$ converges

$\not\Longleftarrow$

$p \Rightarrow q$

$\sum a_k$ diverges $\Longrightarrow$ $\sum |a_k|$ diverges

not p $\Rightarrow$ not q

<u>Proof</u>

$$|a_k| = \begin{cases} a_k, & a_k \geq 0 \\ -a_k, & a_k < 0 \end{cases}$$

$0 \leq |a_k| + a_k \leq 2|a_k|$

By assumption, $\sum |a_k|$ converges $\Rightarrow$ $2\sum |a_k|$ also converges

Consider
$$\sum (|a_k| + a_k)$$
and compare it with $2\sum |a_k|$

$|a_k| + a_k \leq 2|a_k|$ and $2\sum |a_k| < \infty$

By Comparison Test, series $\sum (|a_k| + a_k)$ converges.

$$\sum a_k = \sum (a_k + |a_k| - |a_k|) = \underbrace{\underbrace{\sum (a_k + |a_k|)}_{\text{converges}} - \underbrace{\sum |a_k|}_{\text{conv.}}}_{\text{finite \#}}$$

$\therefore$ $\sum a_k$ converges ∎

Determine whether series diverges, converges absolutely or converges conditionally.

Ex $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\underbrace{\frac{1}{\sqrt{k}}}_{a_k}$: alternating series

$$\left.\begin{array}{l} \dfrac{1}{\sqrt{k+1}} < \dfrac{1}{\sqrt{k}} \quad \checkmark \\ \lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty}\dfrac{1}{\sqrt{k}} = 0 \end{array}\right\} \Rightarrow \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}} \text{ converges by alt. series Test}$$

$\displaystyle\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{\sqrt{k}}\right| = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$: diverges as p-series w/ $p=\frac{1}{2}<1$

$\therefore \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k}}$ converges conditionally

Ex $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k^3}} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{3/2}}$: converges by alt. series Test

$\displaystyle\sum_{k=1}^{\infty}\left|(-1)^{k+1}\frac{1}{k^{3/2}}\right| = \sum\frac{1}{k^{3/2}}$: converges as p-series w/ $p=\frac{3}{2}>1$

$\therefore \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k^3}}$ converges absolutely.

$\underline{\underline{Ex}}$ $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$

$\left|\frac{\sin k}{k^2}\right| \le \frac{1}{k^2}$ but $\sum \frac{1}{k^2}$ converges as p-series w/ $p=2$

$\Rightarrow \sum \left|\frac{\sin k}{k^2}\right|$ converges by Comparison Test

$\therefore \sum \frac{\sin k}{k^2}$ converges by Thm 9.21

Hence, $\sum \frac{\sin k}{k^2}$ converges absolutely.

$\underline{\underline{Ex}}$ $\sum \frac{(-1)^k k}{k+1} = \sum (-1)^k \underbrace{\frac{k}{k+1}}_{a_k}$

$\lim_{k\to\infty} \frac{k}{k+1} = \lim a_k = 1 \neq 0$

$\Rightarrow \sum \frac{(-1)^k k}{k+1}$ diverges by Divergence Test