3/24/2014

Chapter 10. Power series

Power series is an infinite series whose terms depend on a variable.

10.1 Approximating functions with polynomials

Def Power series is a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = \underbrace{c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n}_{n^{th} \text{ degree polynomial}} + c_{n+1} x^{n+1} + \ldots$$

or in general

$$\sum_{k=0}^{\infty} c_k (x-a)^k = \underbrace{c_1 + c_2 (x-a) + \ldots + c_n (x-a)^n}_{n^{th} \text{ degree polynomial}} + c_{n+1} (x-a)^{n+1} + \ldots$$

$$\left.\begin{array}{l} c_k: \text{ coefficients} \\ a: \text{ center of the power series} \end{array}\right\} \text{constants}$$

Name "power series" is because this

this series involves powers of x or $x-a$.

We can interporet power series as follows:

$$\left.\begin{array}{ll} \text{degree } 0: & c_0 \\ \text{degree } 1: & c_0 + c_1 x \\ \text{degree } 2: & c_0 + c_1 x + c_2 x^2 \\ \cdot \ - \ - \ - & \\ \text{degree } n: & c_0 + c_1 x + \ldots + c_n x^n \end{array}\right\} \text{polynomials}$$

$$- \ - \ - \ - \ - \ -$$

$$c_0 + c_1 x + \ldots + c_n x^n + \ldots = \sum_{k=0}^{\infty} c_k x^k : \text{ power series}$$

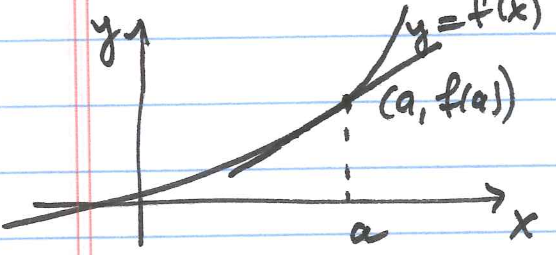
Polynomial approximation

Ex $f(x) = x^7 - 3x^2 + \frac{5}{2}x - 4$: polynomial of degree 7

to evaluate a polynomial at some pt x, we only need to use aritametic operations: addition, subtraction, multiplication, division.

Q How do evaluate logarithmic, exponential, trigonometric functions, square root function (algebraic function)? Clearly, we cannot evaluate these functions using only arithmetic operations. But it turns out that we can approximate such functions by polynomials.

Linear and quadratic approximation



Given a function f, differentiable at $x=a$, we can approximate f by its tangent line in a vicinity of $x=a$.

$$y - f(a) = \underbrace{f'(a)}_{\text{slope}} (x-a)$$: equation of tangent line to $y=f(x)$ at $x=a$

Then

$$\boxed{y = \underbrace{f(a) + f'(a)(x-a)}_{\text{polynomial of degree 1 or linear approximation}} \equiv p_1(x)}$$

We can say that $f(x)$ is approximated by 1st degree polynomial.

Note

$$p_1(a) = f(a), \qquad p_1'(a) = f'(a)$$

i.e. $p_1(x)$ matches f in value and slope at $x=a$.

Linear approximation works well if $f(x)$ is close to a straight line. Can we do better to capture curvature of $f(x)$ as well? Yes. We will include a quadratic term and will require that for this 2nd degree polynomial to match $f(a)$, $f'(a)$ and $f''(a)$.

$$p_2(x) = \underbrace{f(a) + f'(a)(x-a)}_{p_1(x)} + \underbrace{C_2(x-a)^2}_{\text{quadratic term}}$$

Automatically, $p_2(a) = f(a)$

$p_2'(a) = f'(a)$

$$p_2'(x) = f'(a) + 2C_2(x-a)$$

$$p_2''(x) = 2C_2 \underset{\substack{\uparrow \\ \text{has to be}}}{=} f''(a) \Rightarrow C_2 = \frac{1}{2} f''(a)$$

$$\therefore \boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a}$$

We can use these polynomial approxima to evaluate functions like $\ln x$, e^x, $\sin x$, $\sqrt{x}$ etc.

Ex Evaluate $\ln(1.05)$.

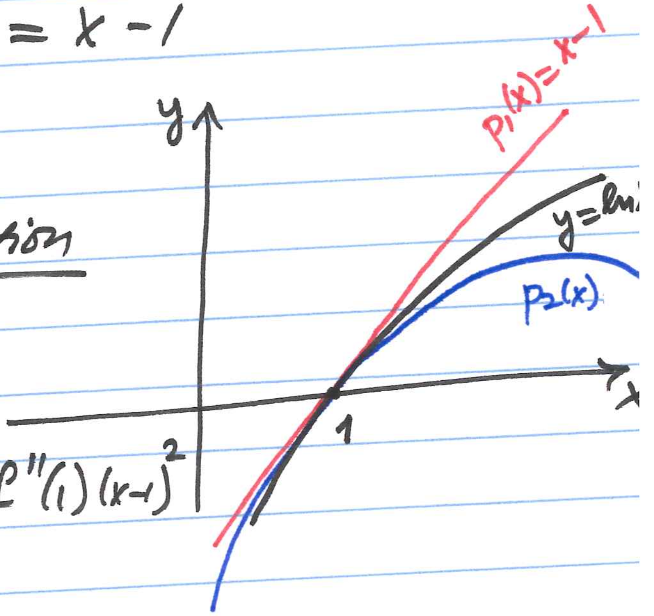
We will use linear and quadratic approximations of $\ln x$ at $x=1$.

1) Linear approximation. $f(x)=\ln x$

$$\ln x \approx p_1(x) = f(1)+f'(1)(x-1)$$

$$f(1)=\ln 1=0, \quad f'(x)=\frac{1}{x}, \quad f'(1)=1$$

$$\therefore \quad p_1(x)=0+1\cdot(x-1)=x-1$$



2) Quadratic approximation

$$\ln x \approx p_2(x) =$$

$$= \underbrace{f(1)+f'(1)(x-1)}_{p_1(x)} + \frac{1}{2}f''(1)(x-1)^2$$

$$f''(x)=-\frac{1}{x^2} \Rightarrow f''(1)=-1$$

$$\therefore \quad p_2(x) = \underbrace{x-1}_{p_1(x)} - 1\cdot\frac{1}{2}(x-1)^2 = (x-1)-\frac{1}{2}(x-1)$$

To approximate $\ln(1.05)$, we will use

$p_1(x)$ and $p_2(x)$ and evaluate them at $x=1.05$.

$\ln(1.05) = 0.04879$: "exact"

and

$$p_1(x) = (x-1)\Big|_{x=1.05} = 0.05$$

$$p_2(x) = \left((x-1) - \frac{1}{2}(x-1)^2\right)\Big|_{x=1.05} = \underset{\text{closer to exact value}}{0.04875}$$

Taylor Polynomials

The above procedure can be extended to higher degree polynomials. For example,

$$p_3(x) = p_2(x) + C_3(x-a)^3$$

Automatically, $p_3(x)$ satisfies conditions

$$p_3(a) = f(a), \quad p_3'(a) = f'(a), \quad p_3''(a) = f''(a)$$

$$p_3'''(x) = C_3 \cdot 3 \cdot 2 \cdot 1 = 3! \cdot C_3$$

$$\underset{\substack{\shortparallel \\ 3!\,C_3}}{p_3'''(a)} = f'''(a) \Rightarrow \boxed{C_3 = \frac{1}{3!} f'''(a)}$$

Hence, cubic polynomial approximation of $f(x)$ is

$$p_3(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2}_{= p_2(x)} + \underbrace{\frac{1}{3!} f'''(a)(x-a)^3}_{\text{cubic term}}$$

We can generalize this procedure to n^{th} degree polynomial:

$$\boxed{p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \ldots \ldots \frac{1}{n!} f^{(n)}(a)(x-a)^n}$$

n^{th} order Taylor polynomial w/ center at $x=a$

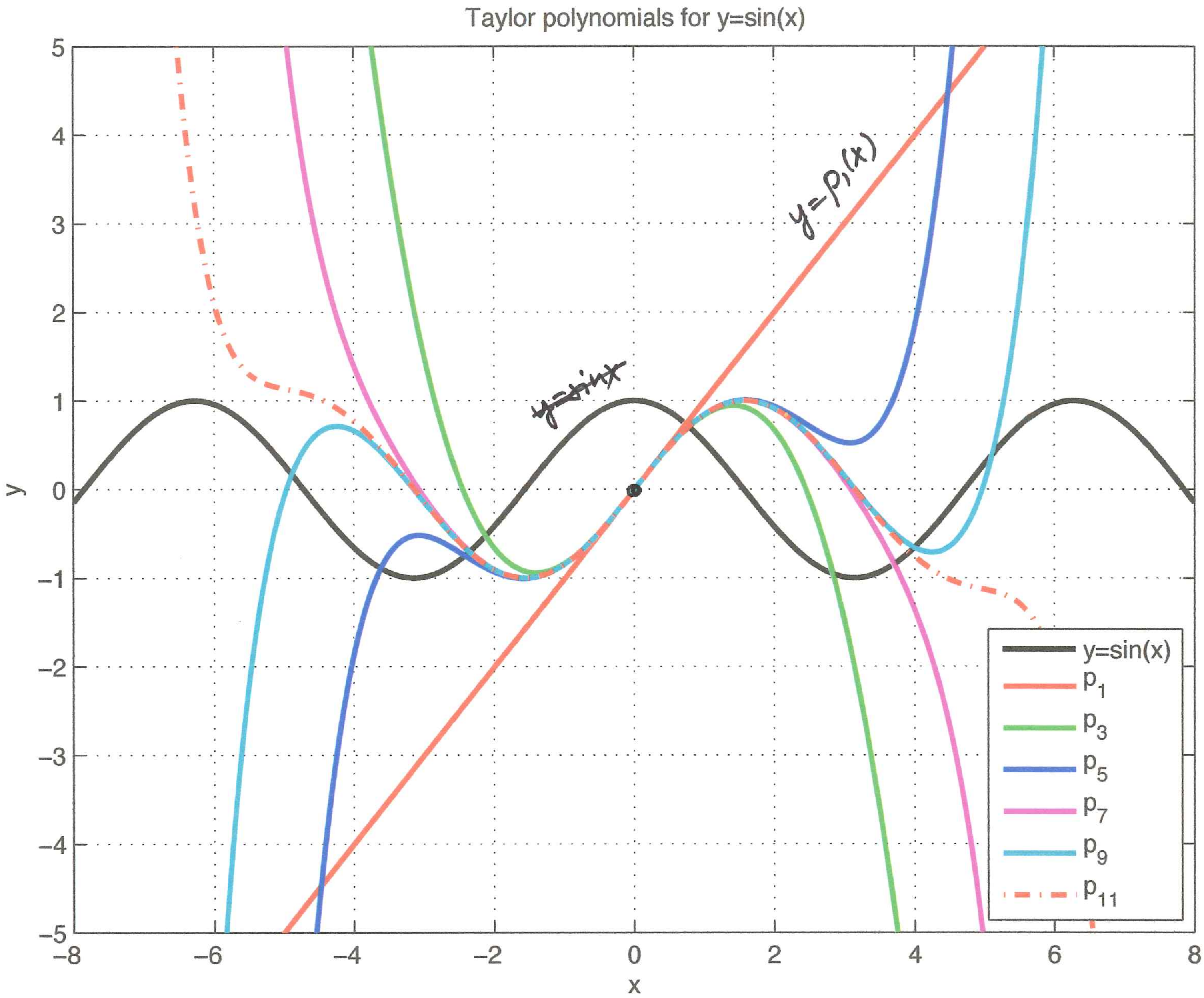
such that

$$p_n(a) = f(a), \quad p_n'(a) = f'(a), \ldots,$$
$$p_n^{(n)}(a) = f^{(n)}(a)$$

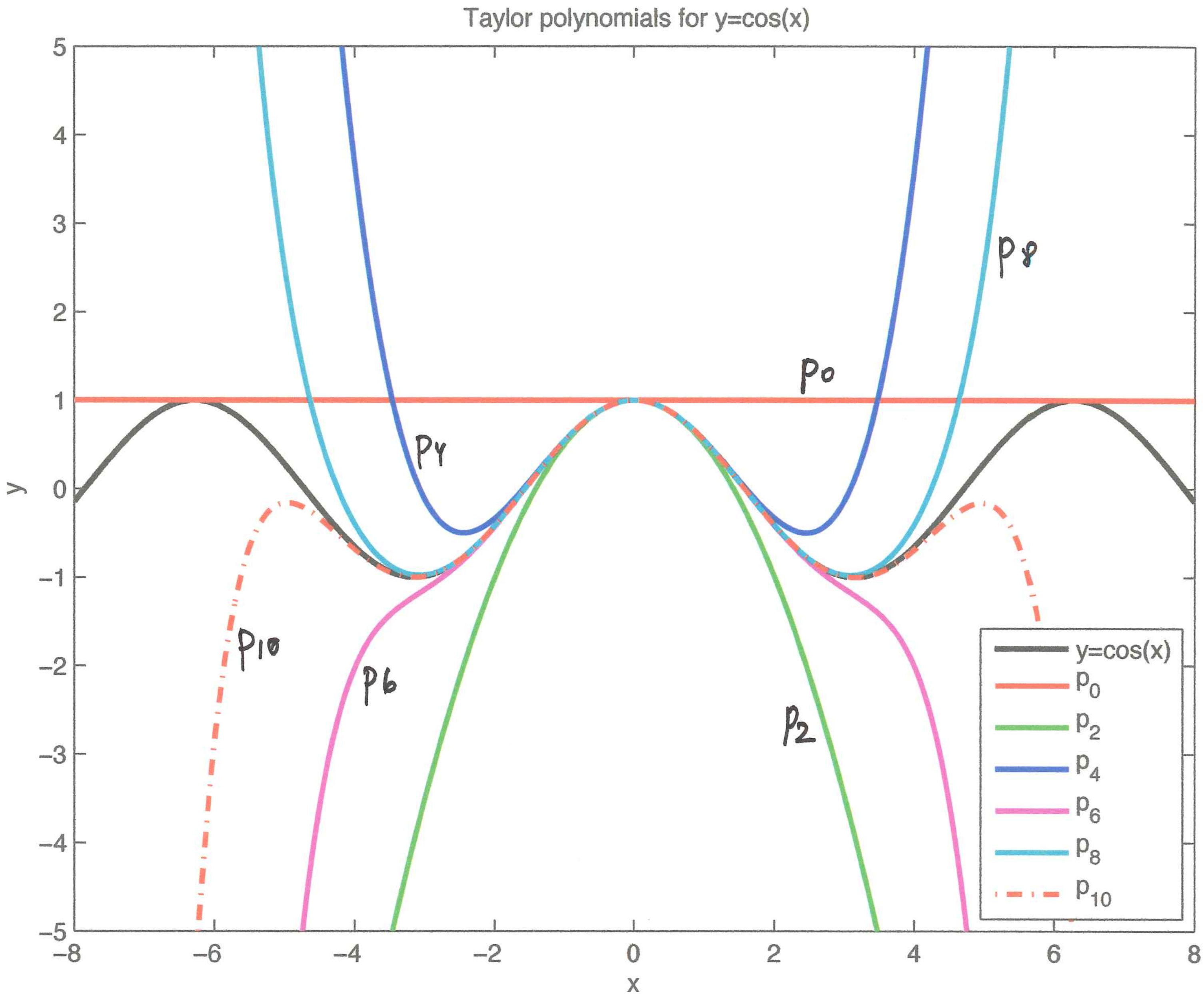
We assume here that $f', f'', \ldots, f^{(n)}$ are defined at $x=a$.

$$p_n(x) = \sum_{k=0}^{n} c_k (x-a)^k, \quad c_k = \frac{f^{(k)}(a)}{k!}$$

compact form $\qquad k=0,1,\ldots,n$



Taylor polynomials for y=sin(x)


Taylor polynomials for y=cos(x)
5
4
3
2
1
0
−1
−2
−3
−4
−5
y
−8
−6
−4
−2
0
2
4
6
8
x
p8
p0
p4
p10
p6
p2
y=cos(x)
p_0
p_2
p_4
p_6
p_8
p_{10}

```
clear all

x=-8:1e-2:8;
y=cos(x);

p0=ones(1,length(x));
p2=1-x.^2/2;
p4=p2+x.^4/factorial(4);
p6=p4-x.^6/factorial(6);
p8=p6+x.^8/factorial(8);
p10=p8-x.^(10)/factorial(10);

figure(1);clf(1)
plot(x,y,'k','Linewidth',2)
hold on
plot(x,p0,'r','Linewidth',2)
plot(x,p2,'g','Linewidth',2)
plot(x,p4,'b','Linewidth',2)
plot(x,p6,'m','Linewidth',2)
plot(x,p8,'c','Linewidth',2)
plot(x,p10,'r-.','Linewidth',2)

legend('y=cos(x)','p_0','p_2','p_4','p_6','p_8','p_{10}',4)
ylim([-5 5])
title('Taylor polynomials for y=cos(x)')
xlabel('x')
ylabel('y')

figure(1)
Taylor=strcat('Taylor_polynomials_cos.eps');
print ('-depsc2', Taylor);

z=sin(x);

p1=x;
p3=p1-x.^3/factorial(3);
p5=p3+x.^5/factorial(5);
p7=p5-x.^7/factorial(7);
p9=p7+x.^9/factorial(9);
p11=p9-x.^(11)/factorial(11);

figure(2);clf(2)
plot(x,y,'k','Linewidth',2)
hold on
plot(x,p1,'r','Linewidth',2)
plot(x,p3,'g','Linewidth',2)
plot(x,p5,'b','Linewidth',2)
plot(x,p7,'m','Linewidth',2)
plot(x,p9,'c','Linewidth',2)
plot(x,p11,'r-.','Linewidth',2)

legend('y=sin(x)','p_1','p_3','p_5','p_7','p_9','p_{11}',4)
ylim([-5 5])
title('Taylor polynomials for y=sin(x)')
xlabel('x')
ylabel('y')
grid on

figure(2)
Taylor_sine=strcat('Taylor_polynomials_sin.eps');
print ('-depsc2', Taylor_sine);
```

3/25/2014

10.1 Approximating functions w/ polynomials (cont'd)

Assume that $f(x)$ has derivatives $f', f'', \ldots, f^{(n)}$ defined at $x=a$.

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

$$\ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$: **nth order Taylor polynomial** for $f(x)$ centered at $x=a$

$p_n(x)$ matches value, slope, curvature and all derivatives of f up to order n at $x=a$:

$$p_n(a) = f(a), \quad p_n'(a) = f'(a), \quad \ldots, \quad p_n^{(n)}(a) = f^{(n)}(a)$$

Ex Taylor polynomial for $\sin x$ centered at $a=0$.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f'' &= -\sin x & f''(0) &= 0 \\ f''' &= -\cos x & f'''(0) &= -1 \\ f^{IV} &= \sin x & f^{IV}(0) &= 0 \end{aligned}$$

derivatives cycle through values $\{0, 1, 0, -1\}$

$$p_n(x) = \overbrace{\underbrace{\cancelto{0}{f(0)} + \underbrace{f'(0)}_{1}\, x}_{p_1} + \cancelto{0}{\frac{f''(0)}{2!}}\, x^2}^{p_2} + \underbrace{\frac{f'''(0)}{3!}}_{-\frac{1}{3!}} x^3 +$$

$$+ \cancelto{0}{\frac{f^{IV}(0)}{4!}} x^4 + \underbrace{\frac{f^{V}(0)}{5!}}_{\frac{1}{5!}} x^5 + \cancelto{0}{\frac{f^{VI}(0)}{6!}} x^6 +$$

$$+ \underbrace{\frac{f^{VII}(0)}{7!}}_{-\frac{1}{7!}} x^7 + \dots + \underbrace{\frac{f^{(n)}(0)}{n!}}_{=\frac{(-1)^k}{(2k+1)!}} x^n$$

$n=2k+1$: odd #

$n=1 \quad f'(0)=1$

$n=3 \quad f'''(0)=-1 \qquad 3=2\cdot1+1 \qquad k=1$

$n=5 \quad f^{V}(0)=1 \qquad 5=2\cdot2+1 \qquad k=2$

$n=7 \quad f^{VII}(0)=-1 \qquad 7=2\cdot3+1 \qquad k=3$

$$f^{(n)}(0) = f^{(2k+1)}(0) = (-1)^k$$

n is odd

$n=2k+1$

$p_1(x) = x$

$p_2(x) = x = p_1(x)$

$p_3(x) = x - \frac{x^3}{3!}$

$p_4(x) = p_3$

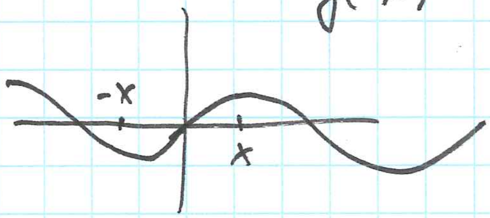
$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$p_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

etc.

$y=\sin x$: odd function

$y(-x) = -y(x)$



all Taylor polynomials are odd functions since they have odd powers of x

$$\boxed{\begin{array}{l} y = \sin x \\ p_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\ n = 2k+1 \end{array}}$$

Ex $f(x) = \cos x$, $a = 0$

$f' = -\sin x$

$f'' = -\cos x$

$f''' = \sin x$

$f^{IV} = \cos x$

$f^{V} = -\sin x$

etc

$f(0) = 1$

$f'(0) = 0$

$f''(0) = -1$

$f'''(0) = 0$

$f^{IV}(0) = 1$

$f^{V}(0) = 0$

derivative cycles through values $\{1, 0, -1, 0\}$

$$p_n(x) = \overbrace{f(0)}^{=1} + \overset{0}{\cancel{f'(0)}}\,x + \overbrace{\frac{f''(0)}{2!}}^{-\frac{1}{2!}} x^2 + \overset{0}{\cancel{\frac{f'''(0)}{3!}}} x^3 +$$

$$+ \overbrace{\frac{f^{IV}(0)}{4!}}^{\frac{1}{4!}} x^4 + \overset{0}{\cancel{\frac{f^{V}(0)}{5!}}} x^5 + \overbrace{\frac{f^{VI}(0)}{6!}}^{=\frac{-1}{6!}} x^6 + \dots$$

$$+ \frac{f^{(n)}(0)}{n!} x^n$$

$n = 2k$: even

$y = f(x) = \cos x$

even powers, p_n is even f^n

$$\boxed{p_n(x) = \underbrace{\overbrace{\underbrace{1}_{p_0} - \frac{x^2}{2!}}^{p_2} + \frac{x^4}{4!}}_{p_4} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \frac{(-1)^k}{(2k)!} x^{2k}}$$

$$p_0(x) = 1$$

$$p_2(x) = 1 - \frac{x^2}{2!}$$

$$p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

etc

Ex $f(x) = e^x$, $a = 0$

$$f' = f'' = \ldots = f^{(n)} = e^x$$

$$\Rightarrow f(0) = f'(0) = \ldots = f^{(n)}(0) = e^0 = 1$$

$$p_n(x) = \underbrace{f(0)}_{1} + \underbrace{f'(0)}_{1} x + \underbrace{\frac{f''(0)}{2!}}_{\frac{1}{2!}} x^2 + \ldots + \underbrace{\frac{f^{(n)}(0)}{n!}}_{\frac{1}{n!}} x^n$$

$$\boxed{p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}} \quad f(x) = e^x$$

Taylor polynomials can be used to approximate functions.

Ex. Approximate $e^{0.1}$. 0.1 is close to $a=0$.

$$e^x \approx p_n(x)$$

$$e^{0.1} \approx p_n(0.1)$$

$$e^{0.1} = 1.1051709 : \text{"exact"}$$

abs. error $= | f(x) - p_n(x) |$

abs.er. $= err_n$
$= | e^{0.1} - p_n(0.1) |$

$n=0$ $\quad p_0(0.1) = 1$ $\qquad err_0 = 1.05 \times 10^{-1}$

$n=1$ $\quad p_1(0.1) = (1+x)\Big|_{x=0.1} = 1.1$ $\qquad err_1 = 5.17 \times 10^{-3}$

$n=2$ $\quad p_2(0.1) = 1.105$ $\qquad err_2 = 1.71 \times 10^{-4}$

$n=3$ $\quad p_3(0.1) = 1.105167$ $\qquad err_3 = 4.25 \times 10^{-6}$

converge to
exact value $e^{0.1} = 1.1051709$

Note as $n \nearrow$, abs. error $| e^{0.1} - p_n(0.1) | \searrow$

Ex Approximate $\sqrt{18}$ $\qquad f(x) = \sqrt{x}$, $a = 16$: close to 18 and $\sqrt{16} = 4$

$f = \sqrt{x} = x^{\frac{1}{2}}$ $\qquad f(16) = \sqrt{16} = 4$

$f' = \frac{1}{2} x^{-\frac{1}{2}}$ $\qquad f'(16) = \frac{1}{2} \frac{1}{\sqrt{16}} = \frac{1}{8}$ $\qquad (x^n)' = n x^{n-1}$

$f'' = \frac{1}{2}(-\frac{1}{2}) x^{-\frac{3}{2}} = -\frac{1}{4} x^{-\frac{3}{2}}$

$f''(16) = -\frac{1}{4} \frac{1}{16^{3/2}} = -\frac{1}{256}$

$f''' = -\frac{1}{4}(-\frac{3}{2}) x^{-5/2} = \frac{3}{8} x^{-5/2}$

$16^{\frac{3}{2}} = (16^{\frac{1}{2}})^3 = 4^3$

etc $\qquad f'''(16) = \frac{3}{8} \frac{1}{(16)^{5/2}} = \frac{3}{8192}$

$$p_n(x) = \overbrace{\underbrace{f(16)}_{p_0} + f'(16)(x-16)}^{p_1} + \frac{f''(16)}{2!}(x-16)^2 +$$

$$+ \frac{f'''(16)}{3!}(x-16)^3 + \ldots + \frac{f^{(n)}(16)}{n!}(x-16)^n$$

$$p_0(x) = 4$$

$$p_1(x) = 4 + \tfrac{1}{8}(x-16)$$

$$p_2(x) = 4 + \tfrac{1}{8}(x-16) - \frac{1}{256 \cdot 2!}(x-16)^2$$

$$p_3(x) = 4 + \tfrac{1}{8}(x-16) - \frac{(x-16)^2}{256 \cdot 2!} + \frac{(x-16)^3}{8192 \cdot 3!}$$

$$\sqrt{18} \approx p_n(18)$$

$\sqrt{18} = 4.242640687119285$: "exact"

abs. error $= |f(x) - p_n(x)| = |\sqrt{18} - p_n(18)|$

$n=0$ $p_0 = 4$ $\quad$ $err_{n=0} = 2.43 \times 10^{-1}$

$n=1$ $p_1(18) = 4.25$ $\quad$ $err_{n=1} = 7.36 \times 10^{-3}$

$n=2$ $p_2(18) = 4.242188$ $\quad$ $err_{n=2} = 4.53 \times 10^{-4}$

$n=3$ $p_3(18) = 4.242676$ $\quad$ $err_{n=3} = 3.51 \times 10^{-5}$

<u>Note</u> : as $n \nearrow$, err $\searrow$

3/27/2014

Remainder in a Taylor polynomial

$R_n(x) = f(x) - p_n(x)$: remainder in a Taylor polynomial approximation

error

exact

approximation

$\Rightarrow \; f(x) = p_n(x) + R_n(x)$

Thms 10.1 and 10.2 Taylor's Thm

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all $x \in I$

$$f(x) = p_n(x) + R_n(x)$$

where

$$p_n(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n :$$

nth degree Taylor polynomial centered at $x = a$

ξ

and

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1},$$ where ξ is a pt between x and a

The remainder $R_n(x)$ satisfies

$$|R_n(x)| = |f(x) - p_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

where M is an upper bound for $f^{(n+1)}(\xi)$:

$$|f^{(n+1)}(\xi)| \leq M$$

Ex $f(x) = \cos x, \quad a = 0$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} =$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

$$f^{(n+1)}(\xi) = \pm \sin \xi \quad \text{or} \quad \pm \cos \xi$$

$$|f^{(n+1)}(\xi)| \leq 1 = M$$

$$\Rightarrow \quad |R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \right| \leq$$

$$\leq 1 \cdot \frac{|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

Hence, for $f(x) = \cos x$, $|R_n(x)| \leq \dfrac{|x|^{n+1}}{(n+1)!}$

Ex Approximate $\cos(0.1)$ using p_{10} and estimate the error.

$x = 0.1$, $n = 10$

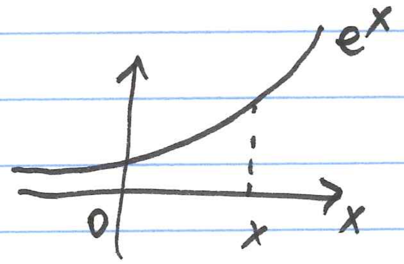
$$|R_{10}(0.1)| \leq \frac{(0.1)^{10+1}}{(10+1)!} \approx 2.5 \times 10^{-9}$$

Ex Estimate $R_n(x)$ for e^x using p_6 centered at $a=0$.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}, \quad \xi \text{ is between } 0 \text{ and } x$$

$$f(x) = e^x \Rightarrow f^{(n+1)}(\xi) = e^{\xi}$$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \right|$$



$$|f^{(n+1)}(\xi)| = e^{\xi} \leq e^x$$

We will approximate $e^{0.45}$. Here $x = 0.45$

$$\Rightarrow |f^{(n+1)}(\xi)| \leq \underbrace{e^{0.45}}_{\text{can't evaluate easily}}$$

in fact, this is what we need to approxim.

Note $e^{0.45} < e^{\frac{1}{2}} < 4^{\frac{1}{2}} = 2 \equiv M$

$e = 2.718\ldots \qquad 0.45 < 0.5 = \frac{1}{2}$

Therefore,

$$|R_n(x)| \le M \cdot \frac{|x|^{n+1}}{(n+1)!} \underset{\substack{\uparrow \\ n=6}}{=} 2\,\frac{(0.45)^{6+1}}{(6+1)!} \approx 1.5 \times 10^{-6}$$

and

$$e^{0.45} \approx p_6(x)\Big|_{x=0.45}$$

Ex How many terms to keep in Taylor polynomial to approximate $f(x) = \ln(1-x)$ w/ error $< 10^{-3}$ on $[-\frac{1}{2}, \frac{1}{2}]$?

$f(x) = \ln(1-x), \quad a = 0 \qquad f(0) = 0$

$$f'(x) = \frac{-1}{1-x}$$

$$f''(x) = \frac{-1}{(1-x)^2}$$

$$f'''(x) = \frac{-2}{(1-x)^3} = \frac{-2!}{(1-x)^3} \qquad 6 = 3\cdot 2\cdot 1 = 3!$$

$$f^{IV}(x) = \frac{-6}{(1-x)^4} = \frac{-3!}{(1-x)^4}$$

$$f^{(n)}(x) = \frac{-(n-1)!}{(1-x)^n}$$

$$\Rightarrow \quad f'(0) = -1, \quad f''(0) = -1, \quad f'''(0) = -2!$$

$$f^{IV}(0) = -3!, \ \dots, \quad f^{(n)}(0) = -(n-1)!$$

$$p_n(x) = \overset{\to 0}{f(0)} + \underbrace{f'(0)x}_{-1} + \frac{\boxed{f''(0)}^{\,=-1}}{2!}x^2 + \frac{\boxed{f'''(0)}}{3!}\underset{-2!}{}x^3 + \dots + \frac{\boxed{f^{(n)}(0)}}{n!}\underset{\to -(n-1)!}{}x^n$$

$$\boxed{p_n(x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots - \frac{1}{n}x^n \qquad \text{for } f(x) = \ln(1-x)}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

$$\boxed{f^{(n)}(x) = \frac{-(n-1)!}{(1-x)^n}}$$

$x \in [-\frac{1}{2}, \frac{1}{2}]$

$\Downarrow$

$\xi \in [-\frac{1}{2}, \frac{1}{2}]$

$$\left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| = \left| \frac{n!}{(1-\xi)^{n+1}} \, \frac{1}{(n+1)!} \right| =$$

$n \to n+1$

$x \to \xi$

$$= \boxed{\frac{1}{|1-\xi|^{n+1}} \cdot \frac{1}{n+1}} \qquad (1)$$

$$|f^{(n+1)}(\xi)| = \left|\frac{n!}{(1-\xi)^{n+1}}\right|$$

$|f^{(n+1)}(\xi)|$ has max when $|1-\xi|^{n+1}$ has min

We have $x \in [-\frac{1}{2}, \frac{1}{2}] \Rightarrow \xi \in [-\frac{1}{2}, \frac{1}{2}]$

$|1-\xi|^{n+1}$ has its min at $\xi = \frac{1}{2} \Rightarrow |1-\xi|^{n+1} = = \left(\frac{1}{2}\right)^{n+1}$

Hence,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}\right| \overset{(1)}{=} \boxed{\frac{1}{|(1-\xi)|^{n+1}} \cdot \frac{|x|^{n+1}}{n+1}}$$

$|x|^{n+1} \le \left(\frac{1}{2}\right)^{n+1}$

$$\le \frac{1}{\cancel{(1-\frac{1}{2})^{n+1}}} \cdot \frac{\cancel{(\frac{1}{2})^{n+1}}}{n+1} = \frac{1}{n+1}$$

$$\therefore \quad |R_n(x)| \le \frac{1}{n+1}$$

We need error $< 10^{-3} \Rightarrow$

$$|R_n(x)| \le \frac{1}{n+1} < 10^{-3}$$

$$\Rightarrow n+1 > 1000 \Rightarrow n > 999$$

for example, we can take $n = 1000$

$n=2$

$\boxed{\#57}$ S10.1

$$\ln(1+x) \approx x - \frac{x^2}{2} \qquad [-0.2,\ 0.2] \qquad a=0$$

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}, \qquad f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2!}{(1+x)^3}, \quad f^{IV} = \frac{-3!}{(1+x)^4}, \ \ldots$$

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2!$$

$$f^{IV}(0) = -3!, \ \ldots$$

$$p_n(x) = x - \frac{x^2}{2} + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \ldots + (-1)^{n-1}\frac{x^n \cdot (n-1)!}{n!}$$

$$p_n(x) = \underbrace{x - \frac{x^2}{2}} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\, x^{n+1}$$

$-0.2 \le x \le 0.2$

$-0.2 \le \xi \le 0.2$

since ξ is between $a=0$ & x

$$\left| f^{(n+1)}(\xi) \right| = \left| \frac{n!}{(1+\xi)^{n+1}} \right| \ \textcircled{\le}$$

$\frac{1}{|1+\xi|}$ has max when $|1+\xi|$ has min and it is at $\xi = -0.2$

$= 3.9062$ when $n=2$

$\frac{2!}{0.8^3}$

$$\boxed{\le} \quad \frac{n!}{(1-0.2)^{n+1}} = \frac{n!}{0.8^{n+1}} = M \qquad |x| \le 0.2$$

$$|R_n(x)| \le |f^{(n+1)}(\xi)| \cdot \frac{|x|^{n+1}}{(n+1)!} \le M \cdot \frac{|x|^{n+1}}{(n+1)!} \le$$

$$\le M \frac{(0.2)^{n+1}}{(n+1)!}$$

$$\le \frac{n!}{(0.8)^{n+1}} \frac{(0.2)^{n+1}}{(n+1)!} = \left(\frac{0.2}{0.8}\right)^{n+1} \cdot \frac{1}{n+1} = \left(\frac{1}{4}\right)^{n+1} \cdot \frac{1}{n+1}$$

can evaluate at $n=2$

$\ln(1+x) \approx x - \frac{x^2}{2}$, $n=2$

$$M \frac{(0.2)^{2+1}}{(2+1)!} = M \frac{(0.2)^3}{3!} =$$

$$= M \frac{0.002}{6} = M \frac{0.001}{3}$$

$\boxed{\#53}$

$$\sin x \approx x - \frac{x^3}{6} \Rightarrow n=3, \quad x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right], \quad a=0$$

$$f(x) = \sin x \qquad |f^{(n+1)}(\xi)| \le 1$$

$$|R_n(x)| = |f^{(n+1)}(\xi)| \cdot \frac{|x|^{n+1}}{(n+1)!} \le \underset{M}{\underset{\shortparallel}{1}} \cdot \frac{|x|^{n+1}}{(n+1)!} \boxed{\le}$$

$$|x| \le \frac{\pi}{4}$$

$$\boxed{\le} \frac{\left(\frac{\pi}{4}\right)^{n+1}}{(n+1)!}$$

$$\Rightarrow |R_n(x)| \le \frac{\left(\frac{\pi}{4}\right)^{n+1}}{(n+1)!} = \frac{\left(\frac{\pi}{4}\right)^{3+1}}{(3+1)!} = 0.0159$$

$$\sin x \approx x - \frac{x^3}{6} \Rightarrow n=3$$

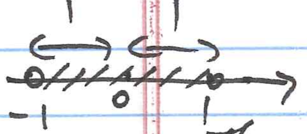
10.2 Properties of Power Series

Recall geometric series

$$\sum_{k=0}^{\infty} r^k = 1+r+r^2+\ldots = \frac{1}{1-r}, \quad |r|<1$$

Replace r with variable x. Then we get a function of x:

$$\sum_{k=0}^{\infty} x^k = 1+x+x^2+\ldots = \frac{1}{1-x}$$ converges if $|x|<1$



or $-1<x<1$

This geometric series converges on $(-1, 1)$.

Def The set of values for which a power series converges is called interval of convergence.

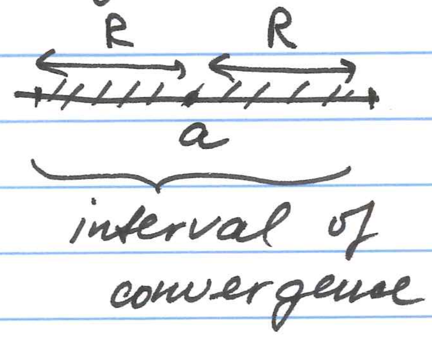
Note Power series can be used to represent functions (exponential, logarithmic, trigonometric etc.)

Consider a power series w/ center a

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

real #5 $\begin{cases} c_k: \text{coefficients}, & c_k: \text{real} \\ a: \text{center}, & a \text{ is real} \end{cases}$ | x is a variable

Def The radius R of convergence of a power series is the distance from center a to the boundary of the interval of convergence.



R: radius of convergence

To determine radius of convergence, we use Ratio or Root Tests.

Recall

Ratio Test $\qquad \sum |a_n|$

$$r = \lim_{k\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

if $r<1$ $\quad \sum |a_n|$ conv.

if $r>1$ $\quad$ diverges

if $r=1$ $\quad$ inconclusive

Root Test

$$\rho = \lim_{k\to\infty} \sqrt[k]{|a_n|}$$

$\rho<1 \Rightarrow \sum |a_n|$ conv.

$\rho>1 \Rightarrow$ diverges

$\rho=1$ $\quad$ inconclusive

Recall

a series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Ex Find interval and radius of convergence.

$$\sum_{k=0}^{\infty} \underbrace{\frac{x^k}{k!}}_{a_k} \qquad a=0$$

Ratio Test:

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\cdot\frac{k!}{x^k}\right| =$$

$$= |x| \lim_{k\to\infty}\frac{1}{k+1} = 0 < 1$$

$\therefore \sum \frac{x^k}{k!}$ converges for all $x \in R$

$(-\infty, \infty)$: interval of convergence

$R=\infty$: radius of convergence

Ex $$\sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k(x-2)^k}{4^k}}_{a_k}$$

Root Test:

$$\rho = \lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\frac{|x-2|^k}{4^k}}$$

$$\sqrt[k]{\dots} = (\ )^{\frac{1}{k}}$$

$= \lim\limits_{k\to\infty} \frac{|x-2|}{4} = |x-2| \, \frac{1}{4} < 1$ for convergence

$\Rightarrow \quad \frac{1}{4}|x-2| < 1$

$|x-2| < 4$ R=4: radius of convergence

or

$-4 < x-2 < 4 \quad \Big| +2$

$-4+2 < x < 4+2$

$-2 < x < 6$: abs. convergence

To check convergence at endpoints, we substitute $x=-2$ and $x=6$ directly in

$\sum\limits_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{4^k}$:

<u>$x=-2$:</u> $\sum\limits_{k=0}^{\infty} \frac{(-1)^k (-4)^k}{4^k} = \sum\limits_{k=0}^{\infty} \frac{(-1)^k (-1)^k \cdot \cancel{4^k}}{\cancel{4^k}} =$

$= \sum\limits_{k=0}^{\infty} (-1)^{2k} = \sum\limits_{k=0}^{\infty} \underbrace{1}_{a_k}$: diverges by Divergence Test $\quad a_k = 1 \not\to 0$

<u>$x=6$</u> : $\sum\limits_{k=0}^{\infty} \frac{(-1)^k (6-2)^k}{4^k} = \sum\limits_{k=0}^{\infty} \underbrace{(-1)^k}_{a_k}$: diverges by Divergence Test $\quad a_k = \pm 1 \not\to 0$

Hence, $(-2, 6)$ is interval of convergence.

Ex $\sum_{k=1}^{\infty} \underbrace{k!\, x^k}_{a_k}$

Ratio Test:

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!\, x^{k+1}}{k!\, x^k}\right| = |x| \underbrace{\lim_{k\to\infty}(k+1)}_{=\infty}$$

$r<1$ only if $x=0$, i.e. series
$(r=0)$

$\sum_{k=1}^{\infty} k!\, x^k$ converges only at one pt $x=0$

(interval of convergence reduces to one pt)

$R=0$: radius of convergence.

Thm 10.3 Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ converges in one of the following ways:

1. Converges for all $x \in \mathbb{R} \Rightarrow (-\infty, \infty)$ is interval of convergence and $R=\infty$

2. There is a finite real # $R>0$: series converges absolutely for $|x-a|<R$ and diverges for $|x-a|>R$.
   R: radius of convergence

endpoints $a \pm R$
need to be checked separately for convergence

3. Series converges only at $x = a$
$\Rightarrow R = 0$

Ex $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{(x-2)^k}{\sqrt{k}}}_{a_k}$ $\qquad a = 2$

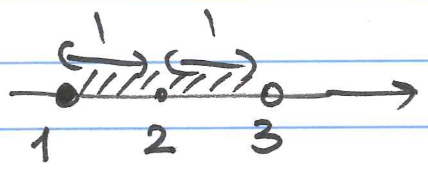
Ratio Test:

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(x-2)^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{(x-2)^k} \right| =$$

$$= |x-2| \lim_{k\to\infty} \sqrt{\frac{k}{k+1}} = |x-2| \sqrt{\lim_{k\to\infty} \frac{k}{k+1}} =$$

$$= |x-2| < 1 \quad \text{for convergence}$$

$\Rightarrow R = 1$ $\qquad |x-a| < R$



Endpoints:

$\underline{\underline{x = 1:}}$ $\displaystyle\sum_{k=1}^{\infty} \frac{(1-2)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$: conv. by altern. Test

$\underline{\underline{x=3}}$ $\sum_{k=1}^{\infty}\frac{(3-2)^k}{\sqrt{k}} = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$: diverges as p-series with $p=\frac{1}{2}<1$

$\therefore$ $x\in[1,3)$: interval of convergence

or $-1\leq x<3$

3/31/2014

Combining Power Series

Power series defines a function of x. We can combine power series to get new functions.

Thm 10.4 Combing power series

Suppose that $\sum c_k x^k$ and $\sum d_k x^k$ converge absolutely to f(x) and g(x) on interval I, ie.

$$f(x) = \sum c_k x^k, \qquad g(x) = \sum d_k x^k$$

Then

1. $\sum (c_k \pm d_k) x^k = f(x) \pm g(x)$: sum & difference

2. $\underbrace{x^m \sum_k c_k x^k}_{f(x)} = \sum_k c_k x^{k+m}$, m : integer, $k+m > 0$: multiplication by power

3. if $h(x) = bx^m$, then

$$\sum c_k (h(x))^k = f(h(x))$$

$$x \to h(x)$$

Ex Recall geom. series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{k=0}^{\infty} x^k, \quad |x| < 1$$

Find

Ex $$\frac{x^5}{1-x} = x^5 \cdot \frac{1}{1-x} = x^5 \sum_{k=0}^{\infty} x^k = \underbrace{\sum_{k=0}^{\infty} x^{k+5}}_{\text{new power series}}$$

conv. for $|x|<1$

Ex $$\frac{1}{1-2x} \underset{\substack{\uparrow \\ x \to 2x}}{=} 1 + 2x + (2x)^2 + (2x)^3 + \dots =$$

$$= 1 + 2x + 4x^2 + 8x^3 + \dots = \sum_{k=0}^{\infty} (2x)^k$$

conv. for $|2x|<1$

or $|x| < \frac{1}{2}$

Ex $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \underset{\substack{\uparrow \\ x \to -x^2}}{=} 1 - x^2 + (-x^2)^2 + (-x^2)^3 + \dots$$

$$= 1 - x^2 + x^4 - x^6 + \dots$$

Thm 10.5 Differentiating and Integrating Power series

Suppose $f(x)$ is defined by its power series $\sum c_k (x-a)^k$ with interval of convergence I, ie.

$$f(x) \underset{\substack{\uparrow \\ \text{converges}}}{=} \sum c_k (x-a)^k, \quad x \in I$$

1. $f(x)$ is continuous on I

2. The power series can be differentiated and integrated term-by-term and results are $f'(x)$ and $\int f(x)\,dx + \underset{\substack{| \\ \text{arb. const}}}{C}$ for all $x \in I$

Ex

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k, \quad |x|<1$$

Differentiate:

$$f'(x) = \frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \ldots =$$

$$= \sum_{k=0}^{\infty} k x^{k-1} = \sum_{k=1}^{\infty} k x^{k-1} \ \boxed{=}$$

Let $l = k-1 \Rightarrow k = l+1$

$k=1 \Rightarrow l = 0$

$k=\infty \Rightarrow l=\infty$

$$\textcircled{=} \sum_{\ell=0}^{\infty} (\ell+1)\, x^{\ell} \underset{\ell \leftrightarrow k}{=} \sum_{k=0}^{\infty} (k+1)\, x^{k}$$

if you like index k

New series $\sum_{k=0}^{\infty} (k+1)\, x^{k}$ conv. abs. for $|x|<1$ by Thm 10.5

at endpoints:

$x=1$: $\sum_{k=0}^{\infty} (k+1)\, 1^{k} = \sum_{k=0}^{\infty} \underbrace{(k+1)}_{a_n \not\to 0}$: diverges by Div. Test

$x=-1$: similarly

$\therefore \sum_{k=0}^{\infty} (k+1) x^{k}$ conv. abs. on $|x|<1$.

Integrate term-by-term.

$$\frac{1}{1-x} = 1+x+x^2+\dots, \qquad |x|<1$$

$$\int \frac{1}{1-x}\, dx = \int (1+x+x^2+\dots)\, dx$$

$|x|<1$
$\Downarrow$
$1-x>0$

$$-\ln|1-x| = x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+C$$

$$-\ln(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+C$$

for all $|x|<1$

Evaluate at any pt in $|x|<1$ to find C:

at $x=0$:

$$-\ln(1-0) = 0 + \frac{0^2}{2} + \dots + C \Rightarrow C=0$$

$$\therefore \; -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

or $$\boxed{\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots, \quad |x|<1}$$

representation of ln(1-x) as a power series

Ex Find power series representation of $\tan^{-1}x$.

Recall $$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

but we found

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \dots, \quad \underset{\Rightarrow |x|<1}{|x^2|<1}$$

$$\int (1-x^2+x^4-x^6+\dots)\,dx = \tan^{-1}x + C$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \tan^{-1}x + C, \quad |x|<1$$

at $x=0$:

$$0 - \frac{0^3}{3} + \dots = \cancelto{0}{\tan^{-1} 0} + C \Rightarrow C=0$$

$$\therefore \quad \boxed{\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \quad |x|<1}$$

$$\tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

at endpoints:

$$x=-1: \quad \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{2k+1}}{2k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \quad \text{conv. altern. series}$$

$$a_k = \frac{1}{2k+1} \to 0 \quad \text{as } k\to\infty$$

$$x=1: \quad \sum_{k=0}^{\infty} \frac{(-1)^k 1^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} : \quad \text{conv. alt. series}$$

Hence, interval of convergence is $[-1, 1]$.

Ex $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ ⊜

we know their power representations

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots , \qquad |x| < 1$$

$$\ln(1+x) = \ln(1-(-x)) \underset{x \to -x}{=} x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\textcircled{=} \left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \dots\right) - \left(-x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \dots\right) =$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

conv. abs. for $|x|<1$

Endpoints

$x=1$: $2\sum_{k=0}^{\infty} \frac{1}{2k+1}$ diverges by Limit Compariso Test

$\sum \frac{1}{k}$ harmonic series diverge

$x=-1$: diverges similarely

10.3 Taylor series

Q Given a function, what is its power series representation?

Suppose f has all order derivatives at $x=a$. Write Taylor polynomial for f w/ center at a and let $n \to \infty$.

We get power series.

$$\overbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n}^{\text{Taylor polynomial of degree } n \quad p_n(x)} +$$

$$+ \underbrace{c_{n+1}(x-a)^{n+1} + \dots}_{\text{remainder}}$$

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, 2, \dots$$

We can write

$$f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} +$$

$$+ \frac{f'''(a)}{3!}(x-a)^3 + \dots$$: Taylor series for f at $x=a$

$a=0$: Maclaurin series

4/1/2014

exam #3 topics

9.4 Divergence and Integral Test
- harmonic series
- p-series
- estimating value of Σ, remainder

9.5 Ratio, Root, Comparison & Limit Comparison Tests

9.6. Alternating Series
- alternating harmonic series
- absolute and conditional convergence

10.1 Taylor polynomials. Power Series
Remainder in Taylor polynomial, estimate

10.2 Properties of Power Series
- convergence of power series (interval, radius of conv.)
- combining power series
- differentiating & integrating power series

10.3 Taylor series (Cont'd)

Assume that function f has derivatives of all orders, $f^{(k)}$, defined on some open interval I containing $x=a$.

(1) $$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+ \\ +\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots$$: Taylor series for f centered at $x=a$

When $a=0$, Taylor series is called Maclaurin series

Q 1. For which x, Taylor series (1) converges?

2. If converges, for which values of x, Taylor series converges to f?

Ex Find Taylor series for $f(x)=\cos x$, $a=0$. Procedure is the same as for Taylor polynomials.

$f(x)=\cos x$ $f(0)=1$

$f'(x)=-\sin x$ $f'(0)=0$

$f''(x)=-\cos x$ $f''(0)=-1$

$2=2\cdot1$

$f'''(x) = \sin x$ $\qquad$ $f'''(0) = 0$

$f^{IV}(x) = \cos x$ $\qquad$ $f^{IV}(0) = 1$ $\qquad$ $4 = 2 \cdot 2$

$\dots$ $\qquad$ $\dots$

$n = 2k$ $\qquad$ $f^{(2k)}(0) = (-1)^k$

even #

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots =$$

(2) $$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n}{(2n)!}x^{2n} + \dots$$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{(2k)!}x^{2k}}_{a_k}$$

Convergence: apply Ratio Test

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2(k+1)}}{\big(2(k+1)\big)!}\cdot\frac{(2k)!}{x^{2k}}\right| \textcircled{=}$$

$$2(k+1) = 2k+2$$

$$\big(2(k+1)\big)! = (2k+2)! = (2k)!\,(2k+1)(2k+2)$$

$$x^{2(k+1)} = x^{2k+2} = x^{2k} \cdot x^2$$

$$\textcircled{=} \lim_{k\to\infty}\left|\frac{\cancel{x^{2k}}\cdot x^2}{\cancel{(2k)!}\,(2k+1)(2k+2)}\cdot\frac{\cancel{(2k)!}}{\cancel{x^{2k}}}\right| =$$

$$= x^2 \lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)} = 0 \Rightarrow r=0<1$$ for all x

∴ Taylor series (2) for $\cos x$ converges absolutely for all $-\infty < x < \infty$.

Ex $f(x) = \dfrac{1}{1-x}$, $a=0$ $\qquad f(0)=1$

$f' = \dfrac{1}{(1-x)^2}$ $\qquad f'(0)=1$

$f'' = \dfrac{2}{(1-x)^3}$ $\qquad f''(0)=2!$

$f''' = \dfrac{2\cdot 3}{(1-x)^4} = \dfrac{3!}{(1-x)^4}$ $\qquad f'''(0)=3!$

. . . . $\qquad$ . . .

$f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$ $\qquad f^{(n)}(0)=n!$

. . . $\qquad$ . . .

$$\Rightarrow c_k = \frac{f^{(k)}(0)}{k!} = \frac{k!}{k!} = 1$$

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots = \sum_{k=0}^{\infty} x^k,\quad |x|<1$$

geometric series

Ex $f(x) = e^x, \quad a = 0$

$$f^{(n)}(x) = e^x, \quad n = 0, 1, 2, \ldots$$

$$f^{(n)}(0) = e^0 = 1 \quad \Rightarrow \quad c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$$

(3) $$\therefore \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots =$$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{x^k}{k!}}_{a_k}$$

Ratio Test:

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| =$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{x^k} \cdot x \cdot \cancel{k!}}{\cancel{k!}\,(k+1) \cdot \cancel{x^k}} \right| = |x| \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$$

$\Rightarrow$ series (3) converges absolutely for all x, i.e. $-\infty < x < \infty$.

The Binomial series

Consider

$$(1+x)^p, \quad p \text{ is integer, } p \geq 0$$

It is a polynomial in x of degree p.

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \dots + \binom{p}{n}x^n$$

where

$$\binom{p}{k} \overset{\text{def}}{=} C_p^k = \frac{p!}{k!\,(p-k)!}$$: **binomial coefficients**

$0! = 1$

$$\binom{p}{0} = 1$$

k, p : integers

$k, p \geq 0$

Ex $(1+x)^2 =$

$p=2$

$$= \binom{2}{0} + \binom{2}{1}x + \binom{2}{2}x^2 \;\text{\textcircled{=}}$$

Aside

$$(a+b)^p = \binom{p}{0}a^p b^0 + \binom{p}{1}a^{p-1}b^1 + \binom{p}{2}a^{p-2}b^2 + \dots + \binom{p}{n-1}a^1 b^{n-1} + \binom{p}{n}a^0 b^n$$

$$\binom{2}{0} = 1$$

$$\binom{2}{1} = \frac{2!}{1!\,(2-1)!} = \frac{2!}{1!} = 2$$

$$\binom{2}{2} = \frac{2!}{2!\,(2-2)!} = 1$$

$$\text{\textcircled{=}}\; 1 + 2\cdot x + 1\cdot x^2$$

Ex

$$(1+x)^5 = \overset{=1}{\binom{5}{0}} + \overset{=5}{\binom{5}{1}}x + \overset{=10}{\binom{5}{2}}x^2 + \overset{=10}{\binom{5}{3}}x^3 + \underset{=5}{\binom{5}{4}}x^4 + \underset{=1}{\binom{5}{5}}x^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

$$\binom{5}{1} = \frac{5!}{1!\,(5-1)!} = 5 = \binom{5}{4};\quad \binom{5}{2} = \frac{5!}{2!\,(5-2)!} = \frac{5!}{2!\cdot 3!} = \frac{\cancel{3!}\cdot 4\cdot 5}{2!\cdot\cancel{3!}} = 10 = \binom{5}{3}$$

$$\binom{p}{k} = \frac{p!}{k!\,(p-k)!} = \frac{\cancel{(p-k)!}\,(p-k+1)\ldots(p-1)\,p}{k!\,\cancel{(p-k)!}}$$

$$\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}$$: valid for p real, k integer, $k \geq 0$

Thm 10.6 Binomial series

For real p, the Taylor series for $f(x)=(1+x)^p$ with center $a=0$ is binomial series

$$\sum_{k=0}^{\infty} \binom{p}{k} x^k = \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!} x^k =$$

$$= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

This series converges for $|x|<1$, possibly at $x=\pm 1$ depending on p. If $p \geq 0$: integer, then binomial series terminates and results in a polynomial of degree p.

Ex $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$

$$c_k = \binom{p}{k} = \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!}$$

$p=\frac{1}{2}$

$$c_0 = \binom{\frac{1}{2}}{0} = 1 = \binom{p}{0} \qquad c_2 = \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = -\frac{1}{8}$$

$$= \frac{p(p-1)}{2!}$$

$$c_1 = \binom{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1!} = \frac{1}{2} = \binom{p}{1} = \frac{p}{1!}$$

$$C_3 = \binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2 \cdot 3} = \frac{1}{16}$$

$$\frac{p(p-1)(p-2)}{3!}$$

$$\therefore \sqrt{1+x} = \underset{\binom{p}{0}}{1} + \underset{\binom{p}{1}}{\frac{1}{2}}x - \underset{\binom{p}{2}}{\frac{1}{8}}x^2 + \underset{\binom{p}{3}}{\frac{1}{16}}x^3 - \dots$$