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Ex (Cont'd) $f(x) = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$ (1)

Truncating this series can be used to approximate values of $f(x)$ for some x.

$f(0.15) = \sqrt{1+0.15} = \sqrt{1.15} = 1.0723805\,2948$ "exact"

We can approximate $\sqrt{1.15}$ by retaining the first terms in (1) and using $p_n(0.15)$

$x = 0.15$

n	$p_n(0.15)$	Note
0	1	as $n \nearrow$,
1	1.075	$p_n(0.15)$ gets
2	1.0721 875	closer to exact value
3	1.07 2 3 98 438	

$$f(x) = \sqrt{1+x} = \underbrace{\overbrace{\underbrace{1}_{p_0} + \frac{1}{2}x}^{p_1} - \frac{1}{8}x^2}_{p_2} + \frac{1}{16}x^3 - \dots \quad (p_3 = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3)$$

$p_0(x) = 1$

$p_1(x) = 1 + \frac{1}{2}x$

$p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2$

$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

. . . .

Convergence of Taylor series

Assume that $f(x)$ has all order derivatives $f^{(n)}$ on an open interval I containing a. By Taylor Thm, we can write

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$: <u>nth order Taylor polynomial</u>

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$: <u>Remainder</u>

ξ is between a and x

<u>Thm 10.7</u>

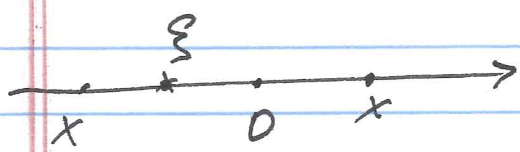
Taylor series for $f(x)$ centered at a converges to $f(x)$ for any $x \in I$ if and only if $\lim_{n\to\infty} R_n(x) = 0$.

<u>Ex</u> Show that Maclaurin series for $f(x) = e^x$ converges to $f(x)$ at all $x \in \mathbb{R}$.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

$f(x) = e^x \Rightarrow f^{(n+1)}(\xi) = e^{\xi}$, ξ is between x and $a=0$



$\Rightarrow |\xi| \leq |x|$

$$|R_n(x)| \leq \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x|^{n+1} = \frac{e^{\xi}}{(n+1)!} |x|^{n+1} \leq$$

$$\leq \frac{e^{|x|}}{(n+1)!} |x|^{n+1}$$

<u>Fix x</u>

$$0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{e^{|x|}}{(n+1)!} |x|^{n+1} = e^{|x|} \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$$

$$= e^{|x|} \lim_{n\to\infty} \frac{\overbrace{|x|\cdot|x| \ldots |x|}^{n+1 \text{ times}}}{1\cdot 2 \ldots n(n+1)} = 0$$

$\therefore \lim_{n\to\infty} R_n(x) = 0$ for all fixed x

Hence, by Thm 10.7, $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges to e^x for all x, i.e.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{any } x \in \mathbb{R}$$

<u>ex</u> $f(x) = \cos x, \quad a=0$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

$$f^{(n+1)}(\xi) = \pm\cos x \text{ or } \pm\sin x$$

$$|f^{(n+1)}(\xi)| \le 1$$

$$\Rightarrow \quad |R_n(x)| \le \frac{|f^{(n+1)}(\xi)|}{(n+1)!}|x|^{n+1} \le \frac{1}{(n+1)!}|x|^{n+1}$$

$$0 \le \left|\lim_{n\to\infty} R_n(x)\right| \le \lim_{n\to\infty}|R_n(x)| \le \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} =$$

$$=0 \Rightarrow \lim_{n\to\infty} R_n(x) = 0 \text{ for all } x$$

(by Thm 10.7)

$\Rightarrow$ Taylor series for $f(x) = \cos x$ converges to $\cos x$ for all x, i.e.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad \forall x$$

10.4 <u>Working with Taylor series</u>

Taylor series can be used to evaluate limits.

Ex Evaluate $\lim\limits_{x\to 0} \dfrac{x^2+2\cos x-2}{3x^4}$

Method I. Use l'Hopital's Rule. (4 times)

$$\lim_{x\to 0}\frac{x^2+2\cos x-2}{3x^4} = \frac{0}{0} = \lim_{x\to 0}\frac{2x-2\sin x}{12x^3} =$$

$$= \lim_{x\to 0}\frac{x-\sin x}{6x^3} = \frac{0}{0} = \lim_{x\to 0}\frac{1-\cos x}{18x^2} = \frac{0}{0} =$$

$$= \lim_{x\to 0}\frac{\sin x}{36x} = \frac{0}{0} = \lim_{x\to 0}\frac{\cos x}{36} = \frac{1}{36}$$

Method II. Expand $\cos x$ in Taylor series about $x=0$.

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6}+\ldots$$

$$\lim_{x\to 0}\frac{x^2+2\cos x-2}{3x^4} = \lim_{x\to 0}\frac{\cancel{x^2}+2\left(\cancel{1}-\cancel{\frac{x^2}{2!}}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)\cancel{-2}}{3x^4}$$

$$= \lim_{x\to 0}\frac{2\left(\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)}{3x^4} = \frac{2}{3}\lim_{x\to 0}\left(\frac{1}{4!}-\frac{x^2}{6!}+\ldots\right)$$

$$= \frac{2}{3}\cdot\frac{1}{4!} = \frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}\cdot 3\cdot 4} = \frac{1}{36}$$

Ex $\lim_{x\to\infty} \left[6x^5 \sin\frac{1}{x} - 6x^4 + x^2\right] =$

Let $t = \frac{1}{x} \Rightarrow$ as $x \to \infty$, $t \to 0$

$$= \lim_{t\to 0} \left[\frac{6}{t^5}\sin t - 6\frac{1}{t^4} + \frac{1}{t^2}\right] =$$

$$= \lim_{t\to 0} \frac{6\sin t - 6t + t^3}{t^5}$$ = Taylor series of $\sin t$ at $t=0$

$$= \lim_{t\to 0} \frac{6\left(\cancel{t} - \cancel{\frac{t^3}{3!}} + \frac{t^5}{5!} - \dots\right) \cancel{-6t} + \cancel{t^3}}{t^5} =$$

$$= \lim_{t\to 0} \frac{6\left(\frac{t^5}{5!} - \frac{t^7}{7!} + \dots\right)}{t^5} = \lim_{t\to 0} 6\left(\frac{1}{5!} - \frac{t^2}{7!} + \dots\right)$$

$$= \frac{6}{5!} = \frac{\cancel{6}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot 4\cdot 5} = \frac{1}{20}$$

Differentiating Power Series

Ex Differentiate Maclaurin series for $f(x) = \sin x$ to show that $f'(x) = \cos x$

$$\frac{d}{dx}(\sin x) = \cos x$$

$\frac{d}{dx}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{d}{dx}(\sin x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots =$$

$$= \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}_{\text{Taylor series for } \cos x\,!} = \cos x$$

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10.4 Working w/ power series (Cont'd)

Ex Find a power series solution of DE

$$y'(t) = y(t) + 2$$

subject to IC $y(0) = 6$.

IC is given at $t=0$. Hence, we expand $y(t)$ in a Taylor series about $a=0$:

$$y(t) = \sum_{k=0}^{\infty} \underbrace{c_k}_{\text{to be determined}} t^k, \qquad c_k = \frac{y^{(k)}(0)}{k!}, \ k=0,1,\dots$$

Our assumption that $y(t)$ has the Taylor series means that $y(t)$ has derivatives at $t=0$ of all orders.

(1) $$y(t) = c_0 + c_1 t + c_2 t^2 + \dots = \sum_{k=0}^{\infty} c_k t^k$$

IC $y(0)=6 \Rightarrow \underset{6}{\underset{\parallel}{y(0)}} = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2 + \dots$

$$\Rightarrow c_0 = 6$$

Differentiate (1) term-by-term to get $y'(t)$:

$$y'(t) = \sum_{k=0}^{\infty} c_k k t^{k-1} = \sum_{k=1}^{\infty} c_k k t^{k-1} =$$

$l = k-1 \Rightarrow k = l+1$

$k=1 \Rightarrow l=0$

$$= \sum_{l=0}^{\infty} c_{l+1}(l+1)t^l \underset{l \leftrightarrow k}{=} \sum_{k=0}^{\infty} c_{k+1}(k+1)t^k$$

Substitute expansions for y and y' into DE

$y' = y + 2$

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)t^k = \sum_{k=0}^{\infty} c_k t^k + 2$$

Note: we have now the same powers of t

t^0: $k=0$: $c_1 \cdot 1 = c_0 + 2 \Rightarrow c_1 = \underset{\overset{\shortparallel}{6}}{c_0} + 2 = 8$

$$\boxed{c_1 = 8}$$

Let $k \geq 1$

t^k: $c_{k+1}(k+1) = c_k$

$$c_{k+1} = \frac{c_k}{k+1}$$

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)t^k = \sum_{k=0}^{\infty} c_k t^k + 2$$

$$\underline{c_1 \cdot 1 \cdot t^0} + c_2 \cdot 2 t^1 + c_3 \cdot 3 t^2 + c_4 \cdot 4 t^3 + \ldots =$$

$$= \underline{c_0 t^0} + c_1 t^1 + c_2 t^2 + \ldots \underline{+ 2}$$

$1 = t^0$: $c_1 \cdot 1 = c_0 + 2$

t^1: $2c_2 = c_1$

t^2: $3c_3 = c_2$

t^3: $4c_4 = c_3$

$\ldots$

t^k: $(k+1)c_{k+1} = c_k$

$$c_{k+1} = \frac{c_k}{k+1} \quad \text{for any } k \geq 1$$

$$\Rightarrow c_k = \frac{c_{k-1}}{k} \qquad c_{k-1} = \frac{c_{k-2}}{k-1}$$

$$\Rightarrow c_{k+1} = \frac{c_k}{k+1} = \frac{c_{k-1}}{(k+1)k} = \frac{c_{k-2}}{(k+1)k(k-1)} = \ldots$$

$1 = k-(k-1)$

$$\ldots = \frac{c_1}{(k+1)k(k-1)\ldots\underbrace{(k-(k-2))}_{=2}} = \frac{c_1}{(k+1)!}$$

Hence, $C_{k+1} = \frac{C_1}{(k+1)!}$ or $\boxed{C_k = \frac{C_1}{k!} = \frac{8}{k!}}$

Then,

$$y(t) = \underset{\overset{\|}{6}}{C_0} + \sum_{k=1}^{\infty} C_k t^k = 6 + \sum_{k=1}^{\infty} \frac{8}{k!} t^k =$$

$$= \underbrace{-2 + 8 + 8\sum_{k=1}^{\infty} \frac{t^k}{k!}} = -2 + 8 \underbrace{\sum_{k=0}^{\infty} \frac{t^k}{k!}}_{e^t} =$$

$$= -2 + 8e^t$$

Hence, $y(t) = -2 + 8e^t$

$y(0) = -2 + 8 = 6$ ✓

Integrating Power Series

Ex <u>Approximate a definite integral</u>

Approximate $\int_0^1 e^{-x^2} dx$ w/ error $\leq 5 \times 10^{-4}$

Expand around $x=0$.

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$ converges for all x

$x \to -x^2$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \frac{(-1)^n x^{2n}}{n!} + \dots$$

converges for all x

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots + \frac{(-1)^n x^{2n}}{n!} + \dots\right) dx$$

$$\int_0^1 e^{-x^2}\,dx = \left(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + \dots\right)\Bigg|_0^1$$

Fundamental Thm of Calculus

$$\int_0^1 e^{-x^2}\,dx = \underbrace{1 - \frac{1}{3} + \frac{1}{5 \cdot 2} - \frac{1}{7 \cdot 3!} + \dots + \frac{(-1)^n}{(2n+1)\,n!} + \dots}_{\text{alternating series}}$$

$$|R_n| \leq a_{n+1} = \frac{1}{(2(n+1)+1)(n+1)!} =$$

$$= \frac{1}{(2n+3)(n+1)!} \leq 5 \times 10^{-4}$$

works for $n \geq 5$

if $n=5 \Rightarrow \quad a_{n+1} = \frac{1}{13 \cdot 6!} \approx 1.07 \times 10^{-4}$

$$\Rightarrow \int_0^1 e^{-x^2}dx \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} \approx$$

$$\approx 0.747$$

$$\sum (-1)^n a_n$$

$(n+1)^{st}$ term is $\frac{(-1)^{n+1}}{\underbrace{(2(n+1)+1)(n+1)!}_{a_{n+1}}} = (-1)^{n+1} a_{n+1}$

$\underline{Ex}$ Use Maclaurin series for $f(x) = \tan^{-1}x$ to evaluate

$$1 - \frac{1}{3} + \frac{1}{5} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + \frac{(-1)^k x^{2k+1}}{2k+1} + \ldots =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

Evaluate at $x=1$:

$$\underbrace{\tan^{-1} 1}_{= \frac{\pi}{4}} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

$$\therefore \; 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

Ex Identify the series

$$\sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!}$$

Recall
$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
converges for all x

$x \to 1-2x$

$$\Rightarrow e^{1-2x} = \sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!}$$
: converges for all $1-2x$
ie, for all x

Ex Please read example about mystery series

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} x^{2k}$$

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ex "Mystery series"

$$\sum_{k=1}^{\infty} \underbrace{\frac{(-1)^k k}{4^k} x^{2k}}_{a_k}$$

Ratio Test

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)\, \overset{x^{2k+2}}{\overset{\parallel}{x^{2(k+1)}}}}{4^{k+1}} \cdot \frac{4^k}{k x^{2k}} \right| =$$

$$= \frac{|x|^2}{4} \underbrace{\lim_{k\to\infty} \frac{k+1}{k}}_{\overset{\parallel}{1}} = \frac{|x|^2}{4} < 1 \text{ for convergence}$$

$$\Rightarrow |x|^2 < 4 \text{ or } |x| < 2$$

endpoints: $x = \pm 2$: divergence

$\therefore$ $(-2, 2)$: interval of convergence.

$$\sum_{k=1}^{\infty} \frac{(-1)^k k x^{2k}}{4^k} =$$

if we had no k, this would be a geometric series.

Note : $\frac{d}{dx}(x^{2k}) = 2k x^{2k-1}$

$\Rightarrow$ our series may be related to derivative of geom. series.

$$= \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2k}{2 \cdot 4^k} \cdot x^{2k-1} \cdot x =$$

$$= \frac{x}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} \underbrace{2k \cdot x^{2k-1}}_{\frac{d}{dx}(x^{2k})} = \frac{x}{2} \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k \frac{d}{dx}(x^{2k})$$

$$= \frac{x}{2} \frac{d}{dx} \sum_{k=1}^{\infty} \left(-\frac{1}{4}\right)^k \cdot x^{2k} = \frac{x}{2} \frac{d}{dx} \underbrace{\sum_{k=1}^{\infty} \left(-\frac{x^2}{4}\right)^k}_{\text{geometric series w/ } r = -\frac{x^2}{4}} =$$

for convergence we need

$|r| < 1$ ie. $\left|-\frac{x^2}{4}\right| < 1$

we have it

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r| < 1$$

$$\sum_{k=1}^{\infty} r^k = \sum_{k=0}^{\infty} r^k - \underbrace{1}_{\text{term when } k=0}$$

$$= \frac{x}{2} \frac{d}{dx} \left[\frac{1}{1-\left(-\frac{x^2}{4}\right)} - 1 \right] = \frac{x}{2} \frac{d}{dx} \left[\frac{1}{1+\frac{x^2}{4}} - 1 \right] =$$

$$= \frac{x}{2} \frac{d}{dx} \left[\frac{4}{4+x^2} - 1 \right] = \frac{x}{2} \frac{d}{dx} \left(\frac{\cancel{4} - (\cancel{4} + x^2)}{4+x^2} \right) =$$

$$= \frac{x}{2} \frac{d}{dx} \left(-\frac{x^2}{4+x^2} \right) = -\frac{x}{2} \cdot \frac{1}{(4+x^2)^2} \left(2x(4+\cancel{x^2}) - \right.$$

$$- 2x\cdot \cancel{x^2}\Big) = -\frac{x}{2}\cdot\frac{8x}{(4+x^2)^2} = -\frac{4x^2}{(4+x^2)^2}$$

Hence,

$$\sum_{k=1}^{\infty}\frac{(-1)^k k}{4^k}x^{2k} = -\frac{4x^2}{(4+x^2)^2}$$ converges on $(-2, 2)$

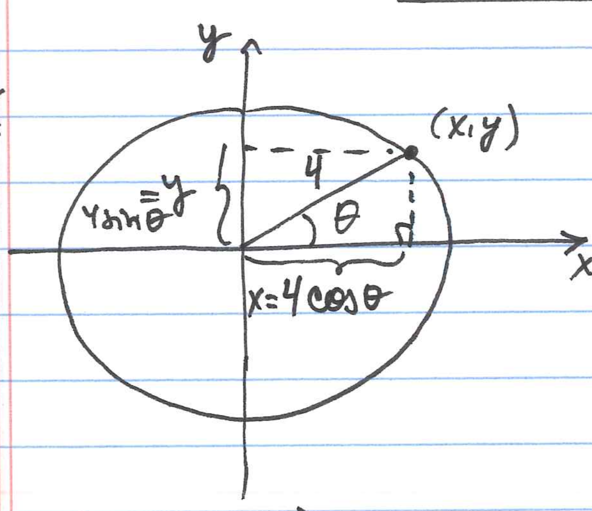
Ch 11 Parametric and Polar Curves

We know that a function can be written as $y=f(x)$: Cartesian form

We will learn about parametric eq$^{\underline{ns}}$
polar coordinates
cylindrical and spherical coordinates

11.1 Parametric equations

Ex



Describe a counterclockwise motion around a circle of rad. 4 mi, competing one lap every hour at a constant speed.

$(x(t), y(t))$: path, $t \geq 0$

angle θ increases by 2π every hour

at $t=0$, $\theta=0$ $\Rightarrow$ $\theta = 2\pi t$, $t \geq 0$

$$x = 4\cos\theta = 4\cos 2\pi t$$
$$y = 4\sin\theta = 4\sin 2\pi t \qquad t \geq 0$$

i.e.

$$\boxed{x = 4\cos 2\pi t, \quad y = 4\sin 2\pi t, \quad t \geq 0}$$

This is an example of <u>parametric equations</u>

$y=f(x)$

$$x \xrightarrow{f} y = f(x)$$ <u>standard two-variable description</u>

$$t \xrightarrow{g} x = g(t)$$
$$t \xrightarrow{h} y = h(t)$$
$\Big\}$ <u>parametric description</u>

In general, <u>parametric equations</u> can be written as

$$x = g(t), \quad y = h(t), \qquad a \leq t \leq b$$

t: <u>parameter</u>

<u>Parametric curve</u> described by these equations consists of all points of the form $(x,y) = (g(t), h(t))$, $\quad a \leq t \leq b$

Ex Parametric parabola

$x = g(t) = 2t, \quad y = h(t) = \frac{1}{2}t^2 - 4, \qquad 0 \le t \le 8$

$t=0: \quad x=0, \quad y=-4 \qquad (0,-4)$

$t=1: \quad x=2, \quad y=\frac{1}{2}-4=-\frac{7}{2} \qquad (2,-\frac{7}{2})$

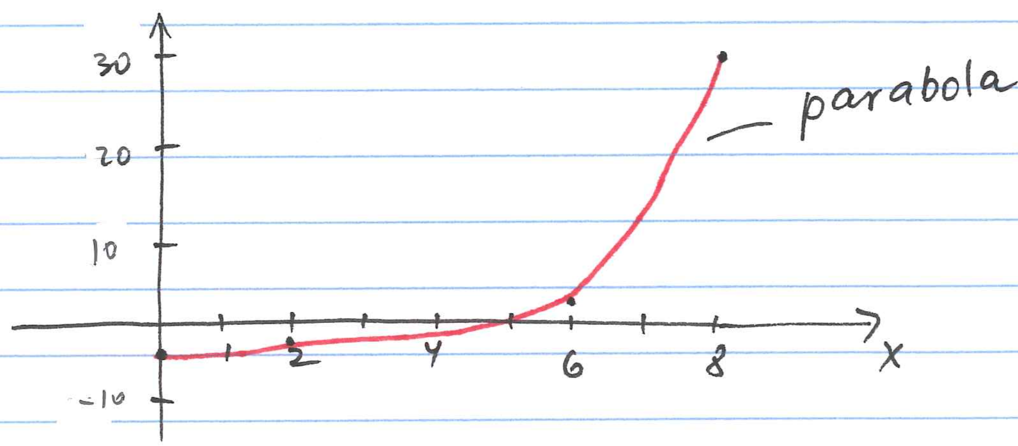
$\cdots$

$t=3: \quad x=6, \quad y=\frac{1}{2}3^2-4=\frac{9}{2}-4=\frac{9-8}{2}=\frac{1}{2} \qquad (6,\frac{1}{2})$

$\cdots$

$t=6: \quad x=12, \quad y=14 \qquad (12,14)$

$t=8: \quad x=16, \quad y=\frac{1}{2}64-4=32-4=28 \qquad (16,28)$



$x=2t, \quad y=\frac{1}{2}t^2-4$

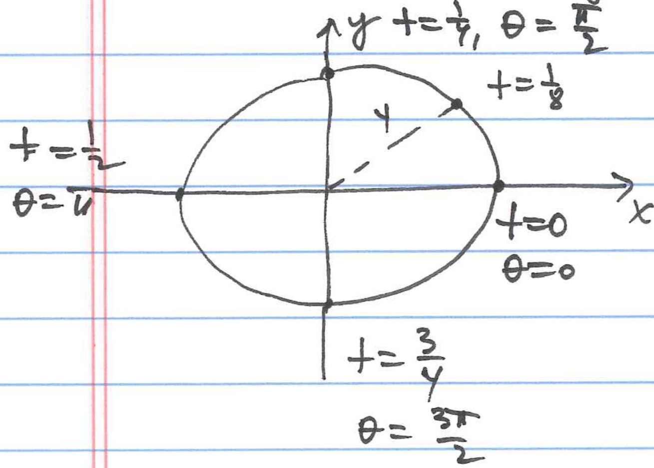
$\Downarrow$

$t=\frac{x}{2} \Rightarrow y=\frac{1}{2}\left(\frac{x}{2}\right)^2-4=\frac{x^2}{8}-4$

$y=\frac{x^2}{8}-4$: parabola

Ex Parametric circle

$x = 4\cos 2\pi t, \quad y = 4\sin 2\pi t, \qquad \underbrace{0 \le t \le 1}_{\text{to describe one lap}}$



$t=0: \; x=4, \; y=0$

$t=\frac{1}{8}: \; x = 4\cos\frac{2\pi}{8} = 4\cos\frac{\pi}{4} = 2\sqrt{2}, \; y = 2\sqrt{2}$

$t=\frac{1}{4}: \; x=0, \; y=4$

$t=\frac{1}{2}: \; x=-4, \; y=0$

$t=\frac{3}{4}: \; x=0, \; y=-4$

$t=1: \; x=4, \; y=0$

$x^2+y^2 = (4\cos 2\pi t)^2 + (4\sin 2\pi t)^2 =$

$= 4^2\left(\cos^2 2\pi t + \sin^2 2\pi t\right) =$

$= 4^2$

$\Rightarrow x^2+y^2$: circle centered at $(0,0)$ w/ radius 4

In general, parametric equations of a circle centered at $(0,0)$ with radius a are

$x = a\cos bt, \quad y = a\sin bt$

a: radius

circle: $x^2+y^2 = (a\cos bt)^2 + (a\sin bt)^2 =$
$= a^2(\cos^2 bt + \sin^2 bt) = a^2$

Function $y=f(x)$ has period T if

$$f(x+T)=f(x)$$

$$\cos(x+2\pi)=\cos x$$

$$\sin(x+2\pi)=\sin x$$

$$\cos(b(x+T)) = \cos(bx+\underbrace{Tb}_{=2\pi}) = \cos bx$$

period

$$\Rightarrow T=\frac{b}{2\pi}$$

b: angular frequency (in radians)

Note: $b>0$: motion is in counterclockwise direction

$b<0$: — " — in clockwise — " —

More generally,

$$x = x_0 + a\cos bt, \qquad y = y_0 + a\sin bt$$

are parametric equations of circle

$$(x-x_0)^2+(y-y_0)^2=a^2$$: circle centered at (x_0, y_0) w/ rad. a

4/10/2014

11.1 Parametric equations

Parametric lines

Parametric equations:

$$\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad -\infty < t < \infty}$$

a, b, x_0, y_0 : const

$a \neq 0$

$$\Downarrow$$

$$t = \frac{x - x_0}{a}$$

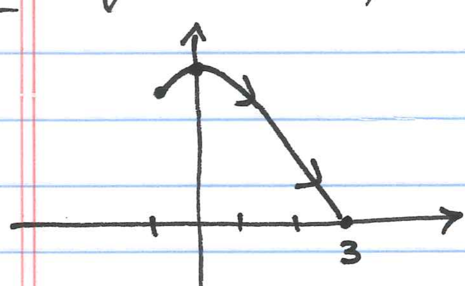
$$\Rightarrow \ y = y_0 + b \cdot \frac{x - x_0}{a}$$

$$y = y_0 + \frac{b}{a}(x - x_0)$$

$$y - y_0 = \frac{b}{a}(x - x_0)$$: line through (x_0, y_0) with slope $\frac{b}{a}$

Parametric equations of curves

Ex $y = 9 - x^2$, $-1 \le x \le 3$: segment of parabola



$y = f(x)$

Let $x = t$

$y = 9 - t^2$ $\quad -1 \le t \le 3$

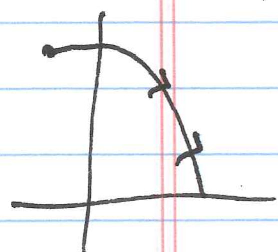
these are parametric eq$^{\underline{ns}}$ of this segment of parabola

Parametric equations are not unique.
For example,

let $x = t+2$ $\qquad -1 \le t+2 \le 3$

$y = 9-(t+2)^2$ $\qquad \Downarrow$

$\qquad -3 \le t \le 1$

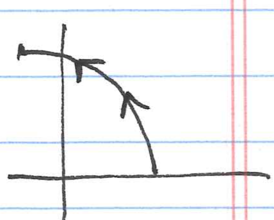


or we can write another parametrization

$x = -t+1$ $\qquad -2 \le t \le 2$

$y = 9-(-t+1)$ $\qquad -2 \le -t \le 2$

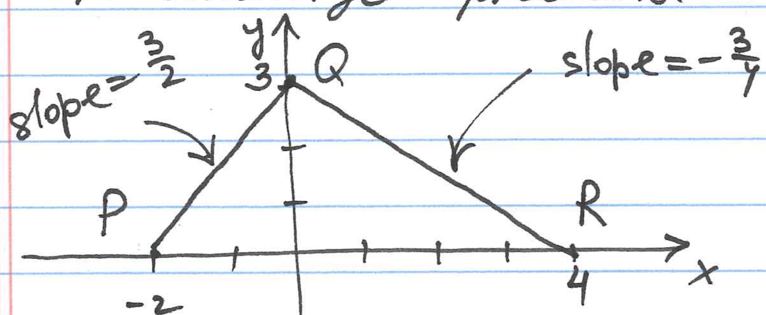
$\qquad -1 \le -t+1 \le 3$



$t=-2 \Rightarrow x = -(-2)+1 = 3$

$t=2 \Rightarrow x = -2+1 = -1$

<u>Ex</u> Parametrize piecewise linear curve:



I-st segment $\qquad x = x_0 + at = x_0 + 2t$ $\qquad 0 \le t \le t_1$

$\qquad y = y_0 + bt = y_0 + 3t$

at $t=0$ $\quad x=-2,\ y=0$

$x=-2 \Rightarrow -2 = x_0 + 2\cdot 0 \Rightarrow x_0 = -2$

$y=0,\ t=0 \Rightarrow \quad 0 = y_0 + 3\cdot 0 \Rightarrow \quad y_0 = 0$

$$\Rightarrow \left.\begin{aligned} x &= -2+2t \\ y &= 3t \end{aligned}\right)$$

at $t=t_1$, $\quad x=0,\ y=3 \Rightarrow \quad 0=-2+2t_1 \Rightarrow t_1 = 1$

check: $y=3$: $\quad 3 = 3\cdot 1$ ✓

Ist segment $\Rightarrow \quad x=-2+2t,\quad y=3t,\quad 0 \le t \le 1$

$\underline{\text{II}}^{nd}$ segment $\quad 1 \le t \le t_2$

at $t=1 \quad x=0,\ y=3$

at $t=t_2 \quad x=4,\ y=3$

slope $= -\frac{3}{4} \Rightarrow \quad x = x_0 + 4t$

$y = y_0 - 3t$

at $t=1 \quad x=0 \Rightarrow \quad 0 = x_0 + 4\cdot 1 \Rightarrow x_0 = -4$

$\Rightarrow x = -4+4t$

at $t=t_2 \quad x=4 \Rightarrow$

$4 = -4 + 4t_2 \Rightarrow t_2 = 2$

at $t=1 \quad y=3 \Rightarrow \quad 3 = y_0 - 3\cdot 1 \Rightarrow y_0 = 6$

at $t=2 \quad y=0 \rightarrow \quad y = 6-3t$

$0 = 6 - 3\cdot 2$ ✓

$\underline{\text{II}}^{nd}$ segment $\left(\begin{aligned} x &= -4+4t \\ y &= 6-3t \end{aligned}\right. \quad 1 \le t \le 2$

4/11/2014

Derivatives and Parametric Equations

Curve $y = f(x)$, then the slope of the tangent line at (x,y) is $\frac{dy}{dx} = f'(x)$

What should we do if a curve is defined by parametric equations

$$x = g(t), \quad y = h(t)$$

Note: $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{h'(t)}{g'(t)}$$

Ex Find the slope of tangent line to curve

$$x = t, \quad y = 2\sqrt{t}$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

$$x'(t) = 1, \quad y'(t) = \frac{d}{dt}(2\sqrt{t}) = \frac{1}{\sqrt{t}}$$

$$\overset{\parallel}{\frac{d}{dt}(t)}$$

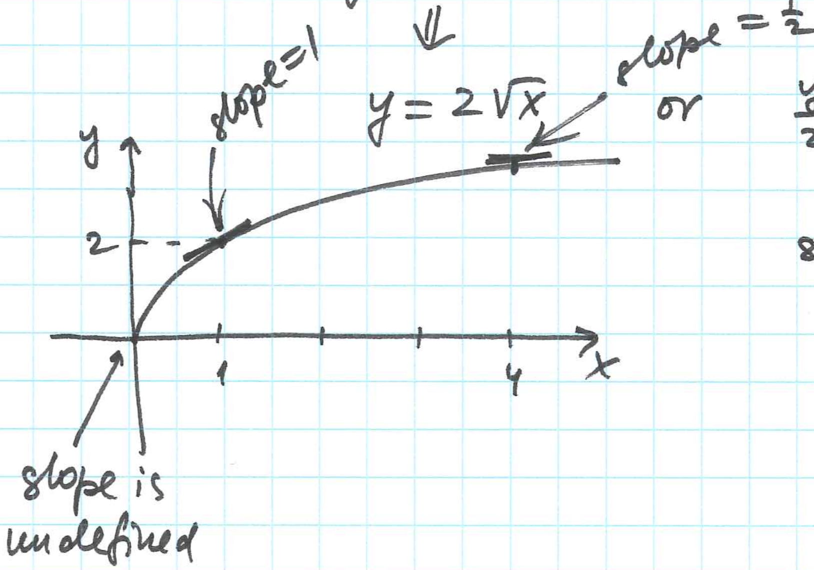
$$\Rightarrow \frac{dy}{dx} = \frac{y'}{x'} = \frac{\frac{1}{\sqrt{t}}}{1} = \frac{1}{\sqrt{t}}$$

as $t \to \infty$, $\frac{dy}{dx} = \frac{1}{\sqrt{t}} \to 0$

if slope of tangent line is zero, the tangent line is horizontal.

$$x = t, \quad y = 2\sqrt{t}$$

$$\Downarrow$$



or $\frac{y}{2} = \sqrt{x} \Rightarrow x = \frac{y^2}{4}$

slope of tangent line at $(1,2)$ is

$$\frac{dy}{dx} = \frac{1}{\sqrt{t}}$$

$x = 1$, $\quad x = t, \; y = 2\sqrt{t}$

$\Downarrow$

$t = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1}} = 1$

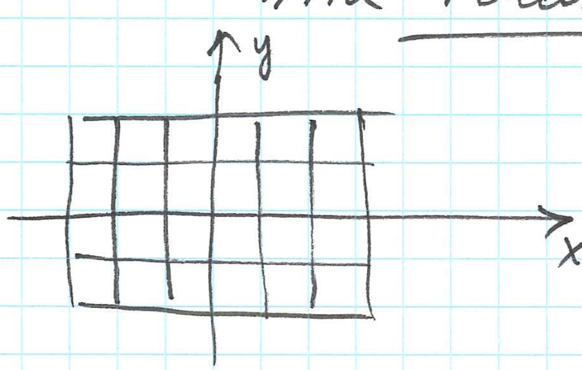
at the origin, $x = y = 0 \Rightarrow t = 0$

$\frac{dy}{dx} = \frac{1}{\sqrt{t}}$: undefined at the origin

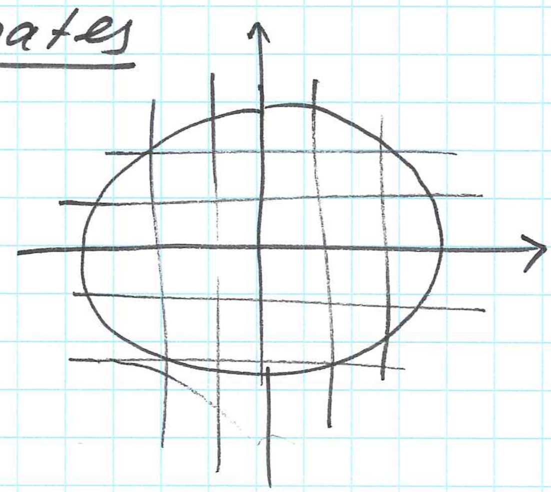
For example, $t = 4 \Rightarrow x = 4, \; y = 2\sqrt{4} = 4$

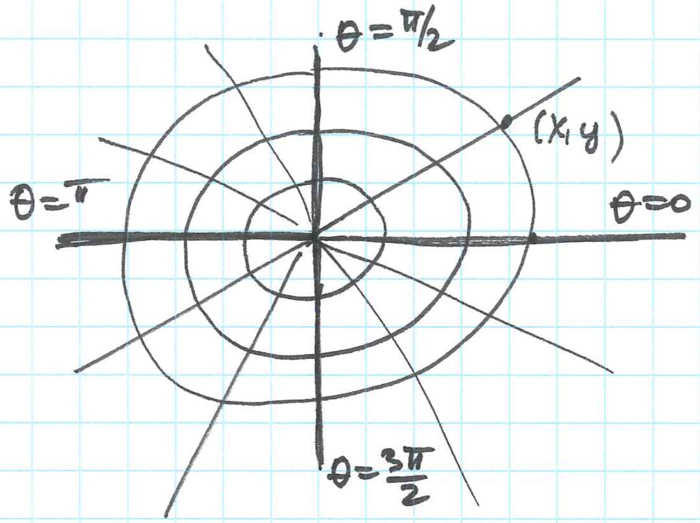
$$\text{slope}\Big|_{(4,4)} = \frac{1}{2} < 1 = \text{slope}\Big|_{(1,2)}$$

11.2 Polar coordinates

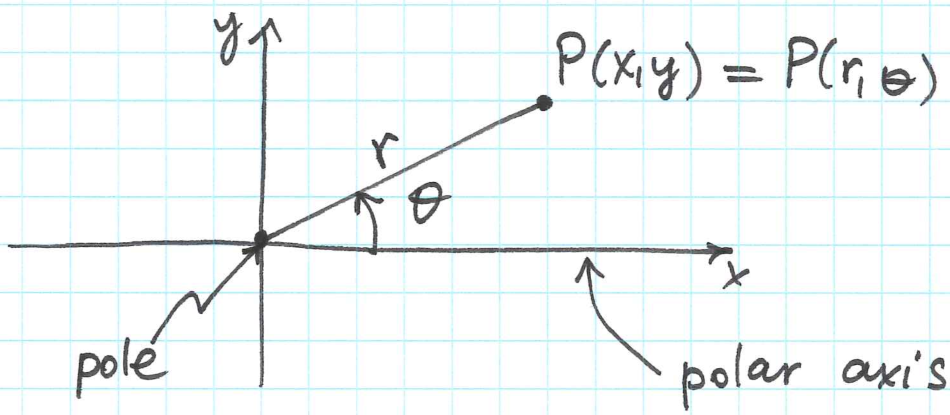


cartesian coordinate system: good for rectangular shapes





Polar coordinate system:
coordinates are constant on circles and rays: good for circular domains.

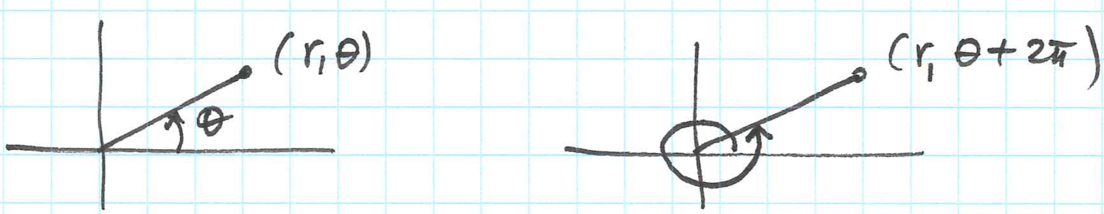


(r,θ): polar coordinates of pt P

r: <u>radial coordinate</u> : signed, directed distance from origin

θ: <u>angular coordinate</u>

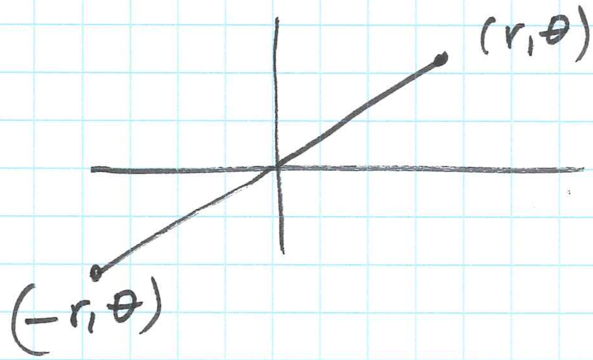
θ is defined up to multiples of 2π



coordinates (r,θ), $(r, \theta \pm 2\pi)$ refer to the same point.

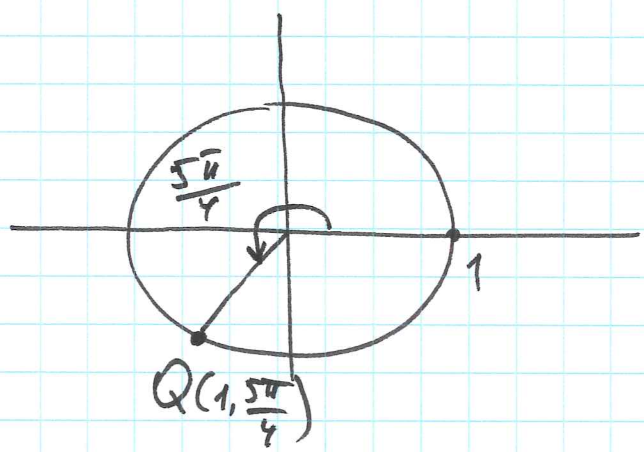
Radial coordinate may be negative.

(r,θ) and $(-r,\theta)$: reflections of each other about the origin

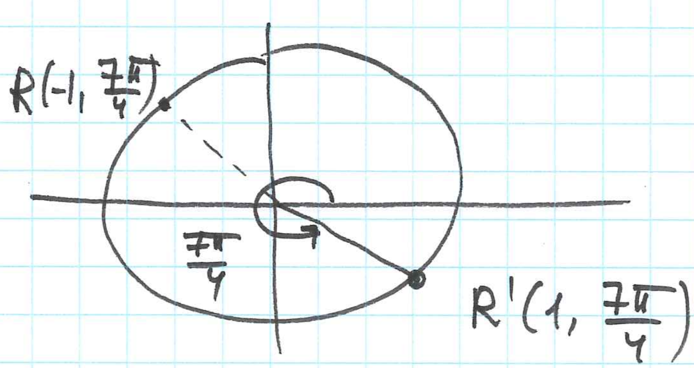


<u>Origin:</u> $r=0$, $\theta=$ any angle

<u>Ex</u> $Q(1, \frac{5\pi}{4})$, $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$



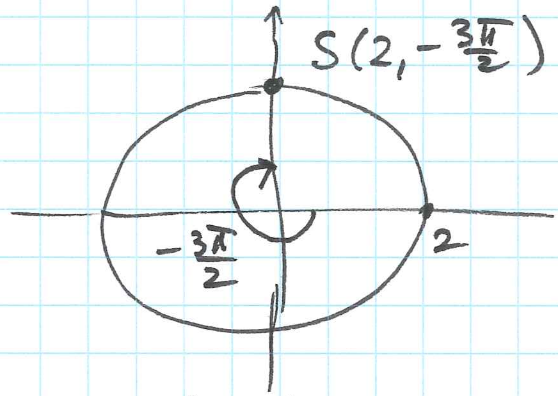
<u>Ex</u> $R(-1, \frac{7\pi}{4})$ is a reflection of $R'(1, \frac{7\pi}{4})$, $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$



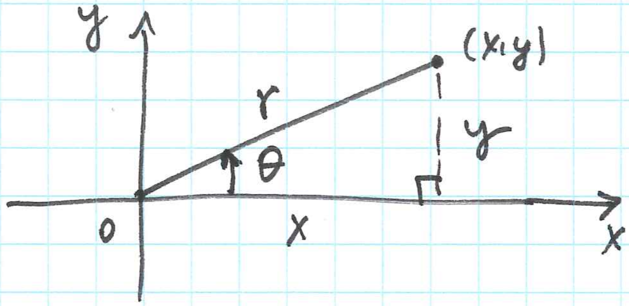
Note

$\theta > 0$ if measured in counterclockwise direction

Ex $S\left(2, -\frac{3\pi}{2}\right)$



Converting between Cartesian and Polar Coordinates



$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}$

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta}$$

$(r,\theta) \to (x,y)$

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$$\Rightarrow \quad \boxed{r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}}$$

We make polar coordinates single-valued by assuming that $r = \sqrt{x^2+y^2}$, $0 \le \theta < 2\pi$

II: $\theta = \arctan\frac{y}{x} + \pi$, $x<0,\ y>0$

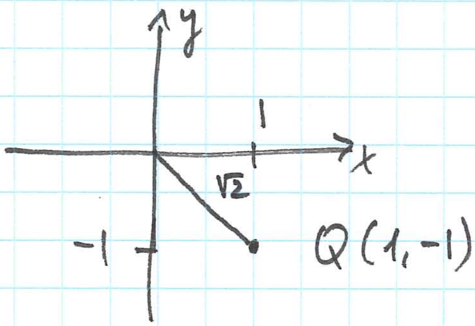
I: $\theta = \arctan\frac{y}{x}$, $x>0,\ y>0$

III: $\theta = \arctan\frac{y}{x} + \pi$, $x<0,\ y<0$

IV: $\theta = \arctan\frac{y}{x} + 2\pi$, $x>0,\ y<0$

$0 \le \theta < 2\pi$

Ex Find polar coordinates of $Q(1,-1)$



$r = \sqrt{2}$, $\theta = \frac{7\pi}{4}$ or $\theta = -\frac{\pi}{4}$ (multivalued)

$Q(1,-1) \Rightarrow x = 1, \quad y = -1$

$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \left(\frac{-1}{1}\right) = (-1)$

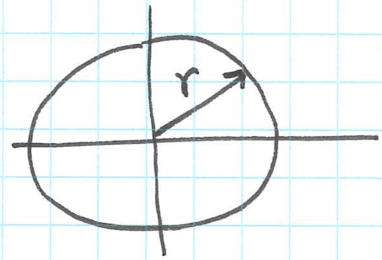
$x = 1 > 0, \ y = -1 < 0 \Rightarrow (1,-1)$ is in IV quadrant

$\theta = \arctan\left(\frac{y}{x}\right) + 2\pi = \arctan\left(\frac{-1}{1}\right) + 2\pi \ =$

$\arctan(-t) = -\arctan t$: odd function

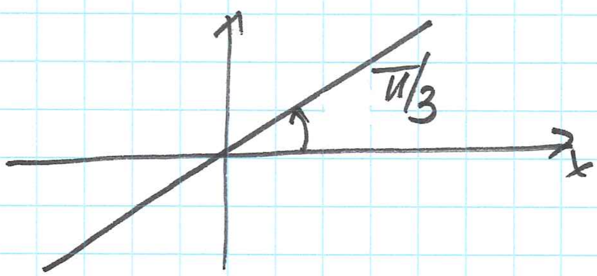
$= -\arctan 1 + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4} \in [0, 2\pi)$

Ex $r=3$: θ is any equation of the circle with radius 3 centered at origin



$r=a$: circle w/ rad. a w/ center at origin

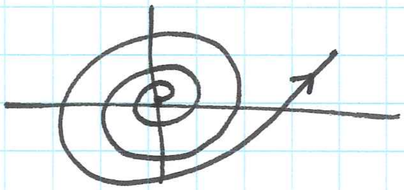
Ex $\theta=\frac{\pi}{3}$ ($\Rightarrow$ r is any) : this is a line if you allow radial coordinate be >0 or <0.



Otherwise, if $r>0$ we have a ray.

11.2 Polar Coordinates (Cont'd)

Ex $r = \theta$: as $\theta \nearrow$, $r \nearrow 0$: spiral in counterclockwise direction



Ex Convert polar equation $r = 6\sin\theta$ to Cartesian coordinate system:

$$r = 6\sin\theta \quad / \cdot r, \; r \neq 0$$

$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$$\underbrace{r^2}_{x^2+y^2} = 6\underbrace{r\sin\theta}_{y}$$

$\therefore$

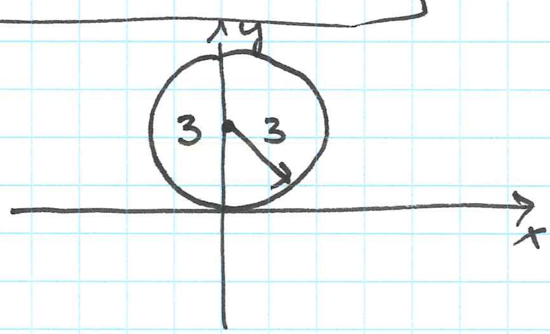
$$x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y = 0$$

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

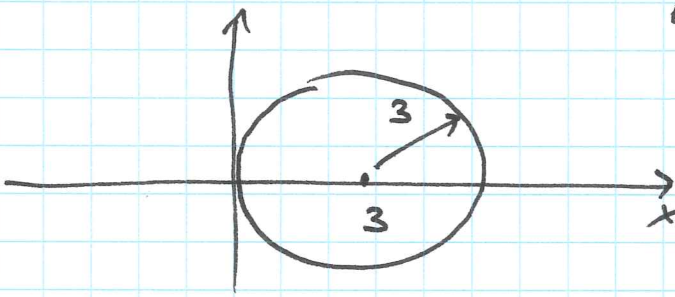
$$x^2 + \underbrace{y^2 - 2\cdot 3y + 3^2}_{} - 3^2 = 0$$

$$\boxed{x^2 + (y-3)^2 = 3^2}$$: circle centered at $(0, 3)$ with radius 3



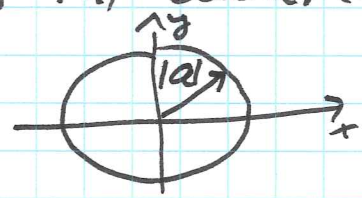
Note: equation $r = 6\cos\theta$ would give

$(x-3)^2 + y^2 = 3^2$: circle centered at $(3,0)$ with radius $r = 3$

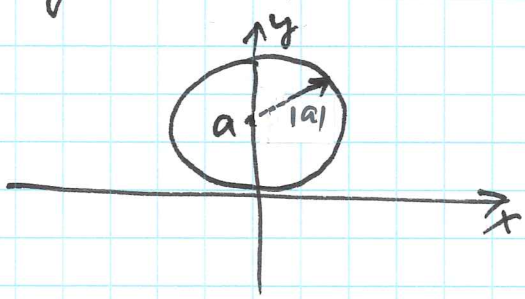


Summary: circles in polar coordinates

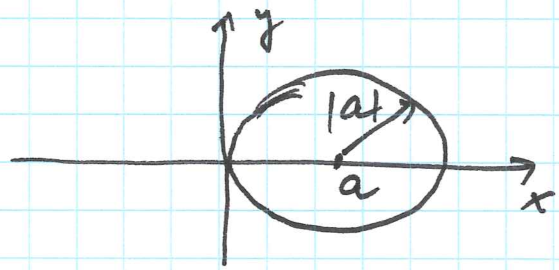
$r = a$: circle of radius $|a|$ centered at origin



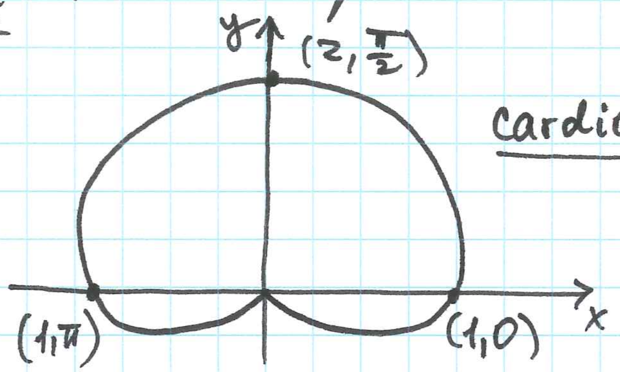
$r = 2a\sin\theta$: circle of radius $|a|$ centered at $(0, a)$



$r = 2a\cos\theta$: circle of radius $|a|$ centered at $(a, 0)$



Ex Plot a polar curve $r = f(\theta) = 1 + \sin\theta$



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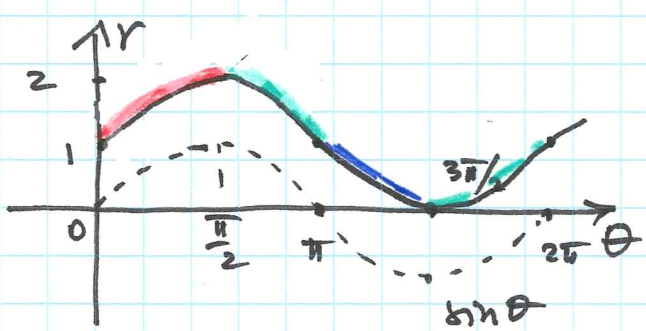
coordinates (r, θ)

A polar curve may be plotted by finding several/many points (r, θ)

It may be time consuming. Besides having several points may not give a clear picture about the shape of the curve.

Another approach is to plot first $r = f(\theta)$ as a function of θ as if r, θ are Cartesian coordinates. Then to find the corresponding shape in polar coordinates.

$r = f(\theta) = 1 + \sin\theta$



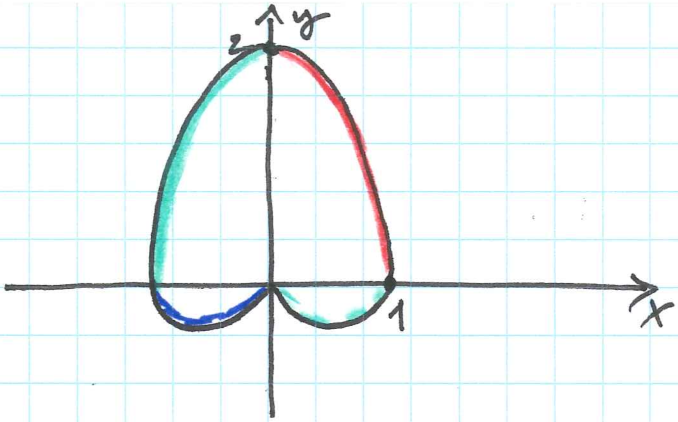
$\theta = 0, \quad r = 1$

$\theta = \frac{\pi}{2}, \quad r = 2$

$\theta = \pi, \quad r = 1$

$\theta = \frac{3\pi}{2}, \quad r = 0$: at the origin

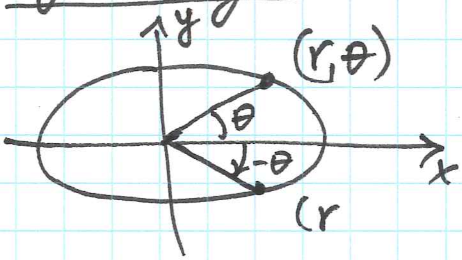
$\theta = 2\pi, \quad r = 1$



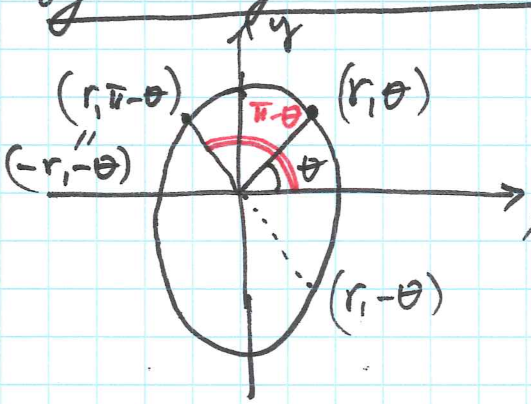
Symmetry in Polar coordinates

Polar curve has

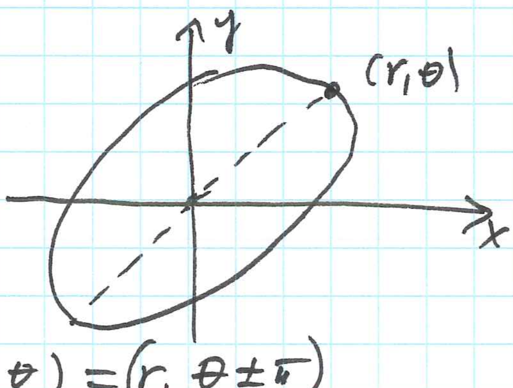
- symmetry about x-axis if point (r,θ) is on the graph whenever $(r,-\theta)$ is on the graph



- symmetry about y-axis if points (r,θ) and $(r,\pi-\theta)=(-r,-\theta)$ are on the graph



- symmetry about the origin



if points (r,θ) and $(-r,\theta)=(r,\theta\pm\pi)$ are on the graph

$(-r,\theta)=(r,\theta\pm\pi)$

11.3 Calculus in Polar Coordinates

Slopes of Tangent lines

If we have a curve $y=f(x)$, then the slope of a tangent line at (x,y) is

$$\frac{dy}{dx} = f'(x)$$

Q What about a curve given in polar form $r=f(\theta)$? It is NOT $f'(\theta)$!

Recall, for a function given by parametric equations

$$x=g(t), \quad y=h(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{h'(t)}{g'(t)}$$

In polar coordinates

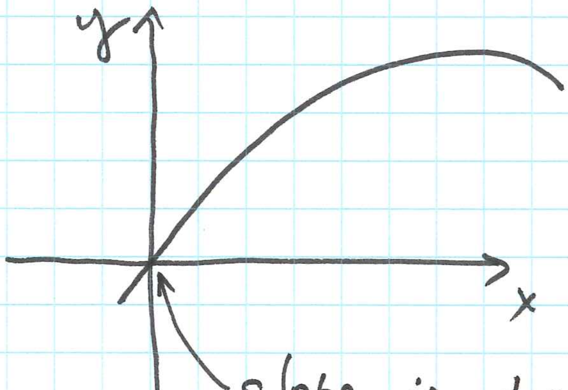
$$x = r\cos\theta = f(\theta)\cos\theta = x(\theta)$$

$$y = r\sin\theta = f(\theta)\sin\theta = y(\theta)$$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}}$$

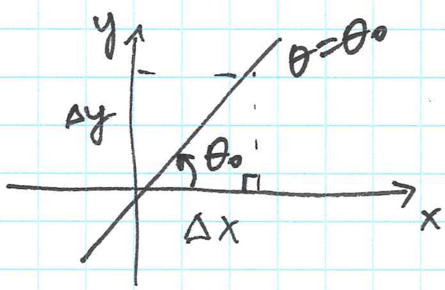
Note if graph of $r = f(\theta)$ passes through the origin $(r=0)$ with some $\theta_0 \Rightarrow f(\theta_0) = 0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta_0} = \frac{f'(\theta_0)\sin\theta_0 + \cancelto{0}{f(\theta_0)}\cos\theta_0}{f'(\theta_0)\cos\theta_0 - \cancelto{0}{f(\theta_0)}\sin\theta_0} = \tan\theta_0$$



slope is $\tan\theta_0$, provided $f'(\theta_0) \neq 0$

Note that line $\theta = \theta_0$



slope $= \frac{\Delta y}{\Delta x}$ = const = $\tan\theta_0$

of line $\theta = \theta_0$

$\Rightarrow$ slope of line $\theta = \theta_0$ is $\tan\theta_0$. Hence, if a curve $r = f(\theta)$ goes through the origin at $\theta = \theta_0$, i.e. $f(\theta_0) = 0$, then tangent line at the origin $(0, \theta_0)$ is $\theta = \theta_0$ w/ slope $\tan\theta_0$.

4/15/2014

Thm 11.2 Slope of a Tangent line

Let f be a differentiable function at θ_0. Then the slope of the line tangent to curve $r = f(\theta)$ at $(f(\theta_0), \theta_0)$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0}$$

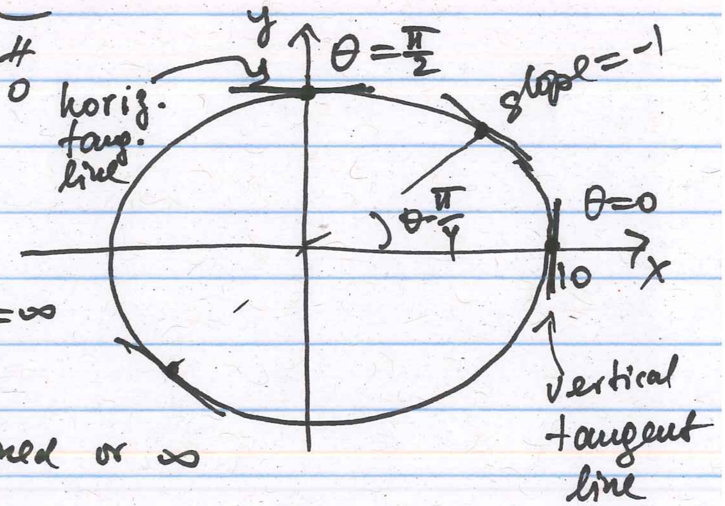
provided that denominator $\neq$ at $\theta = \theta_0$. If at θ_0, $f(\theta_0) = 0$, $f'(\theta_0) \neq 0$, the tangent line is $\theta = \theta_0$ with slope $\tan\theta_0$.

Ex Slopes on a circle

$$r = f(\theta) = 10 \Rightarrow f'(\theta) = 0, \quad f(\theta) \neq 0$$

$$\frac{dy}{dx} = \frac{\overset{0}{f'(\theta)}\sin\theta + \overset{\neq 0}{f(\theta)}\cos\theta}{\underset{0}{f'(\theta)}\cos\theta - \underbrace{f(\theta)}_{\neq 0}\sin\theta} = -\frac{\cos\theta}{\sin\theta} =$$

$$= -\cot\theta$$



$$\theta = 0 \Rightarrow \cot\theta = \left.\frac{\cos\theta}{\sin\theta}\right|_{\theta=0} \Bigg|_{\theta=0} = \frac{1}{0} = \infty$$

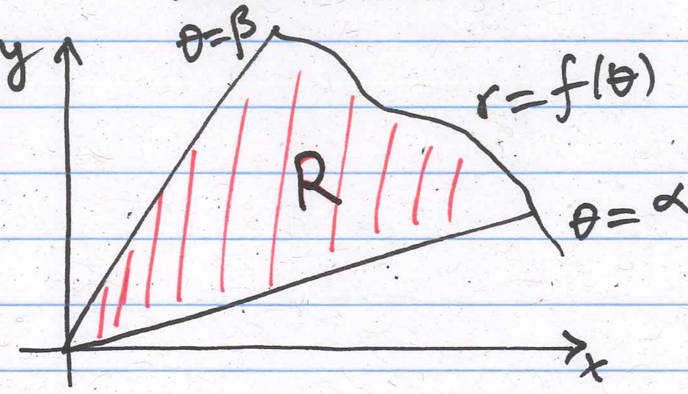
$\Rightarrow$ slope is undefined or ∞

$$\theta = \frac{\pi}{2} \Rightarrow \cot\frac{\pi}{2} = \frac{\cos\pi/2}{\sin\pi/2} = \frac{0}{1} = 0 \Rightarrow \text{slope is } 0$$

$\theta = \frac{\pi}{4} \Rightarrow$ slope $\frac{dy}{dx} = -\cot\frac{\pi}{4} = -1$ ✓

$\theta = \frac{5\pi}{4} \Rightarrow \frac{dy}{dx} = -\cot\frac{5\pi}{4} = -\frac{\cos 5\pi/4}{\sin 5\pi/4} = -\frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$ ✓

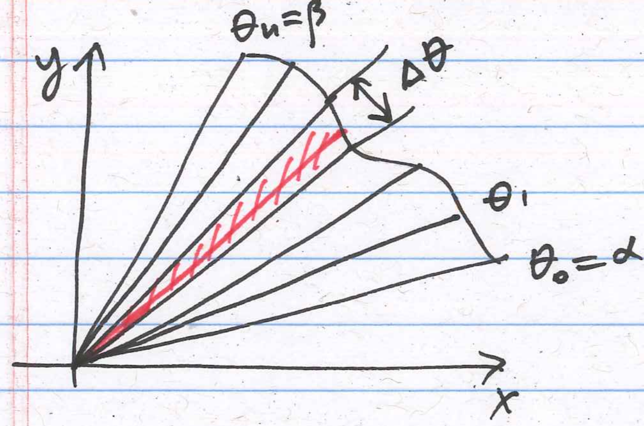
Area of regions bounded by Polar Curves



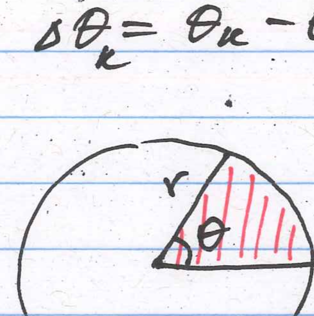
Find area of region R bounded by $\theta=\alpha$, $\theta=\beta$ and $r=f(\theta)$.

Assume f is continuous, $f \geq 0$, $\theta \in [\alpha, \beta]$

Partition: $\alpha = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n = \beta$



$\Delta\theta_k = \theta_k - \theta_{k-1}$



$\bar{\theta}_k$: angle in $[\theta_{k-1}, \theta_k]$

$f(\bar{\theta}_k)$: radial coordinate of "sector"

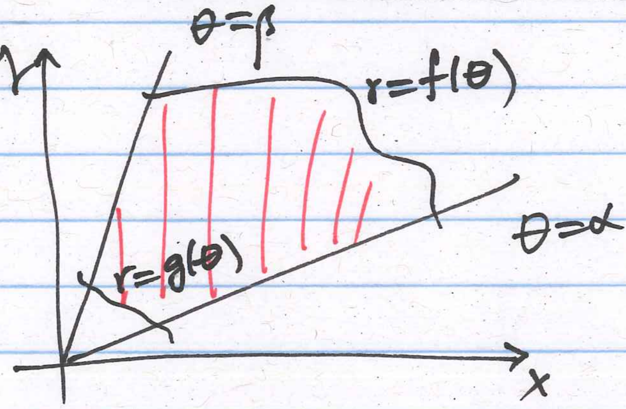
Area of circular sector is $\frac{1}{2}\theta r^2$

Area of a "sector" is $\approx \frac{1}{2}\Delta\theta_k\left(f(\bar{\theta}_k)\right)^2$

$\Rightarrow$ Area of R is $\approx \underbrace{\sum_{k=1}^{n} \frac{1}{2}\Delta\theta_k\left(f(\bar{\theta}_k)\right)^2}_{\text{Riemann sum}} \rightarrow$

$$\xrightarrow[n\to\infty]{} \int_\alpha^\beta \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$

$$\Rightarrow \boxed{\text{Area of } R \text{ is } \frac{1}{2}\int_\alpha^\beta \left(f(\theta)\right)^2 d\theta}$$



assume f and g are continuous, $f(\theta)\geq g(\theta)\geq 0$, $\alpha\leq\theta\leq\beta$

Then area of the region bounded by $r=g(\theta)$, $r=f(\theta)$, $\theta=\alpha$, $\theta=\beta$ is

$$\int_\alpha^\beta \frac{1}{2}\left[\left(f(\theta)\right)^2-\left(g(\theta)\right)^2\right] d\theta$$

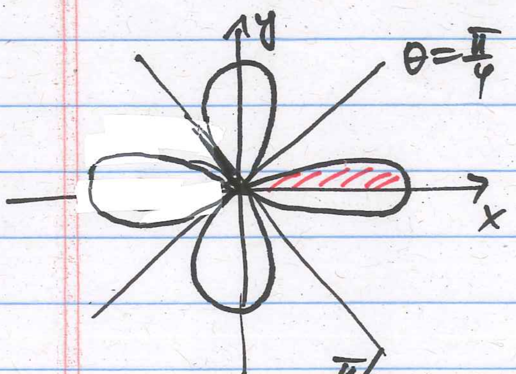
Ex Find area of circle $r=f(\theta)=8$, $0\leq\theta\leq 2\pi$

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}\left(f(\theta)\right)^2 d\theta = \int_0^{2\pi} \frac{1}{2}\cdot 8^2 d\theta =$$

$$= \frac{1}{2}8^2\cdot\theta\Big|_0^{2\pi} = \frac{1}{2}8^2\cdot 2\pi = \pi\cdot 8^2 \checkmark$$

Ex Find the area of the region bounded by

$$r = f(\theta) = 2\cos 2\theta \quad : \text{ has symmetry about } x\text{- and } y\text{-axis}$$



$$\text{Area} = 8\int_0^{\pi/4} \frac{1}{2}\big(f(\theta)\big)^2 d\theta =$$

$$= 8\int_0^{\pi/4} \frac{1}{2}(2\cos 2\theta)^2 d\theta =$$

$$= 8\cdot 2^2 \int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta \, d\theta = 8\cdot 2^2 \frac{1}{2}\int_0^{\pi/4} \frac{1+\cos 4\theta}{2} d\theta =$$

$$= \frac{16}{2}\int_0^{\pi/4} (1+\cos 4\theta)\, d\theta = \frac{16}{2}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/4} =$$

$$= \frac{16}{2}\Big(\frac{\pi}{4} + \frac{1}{4}\underbrace{\sin\pi}_{=0}\Big) = 2\pi \qquad \blacksquare$$

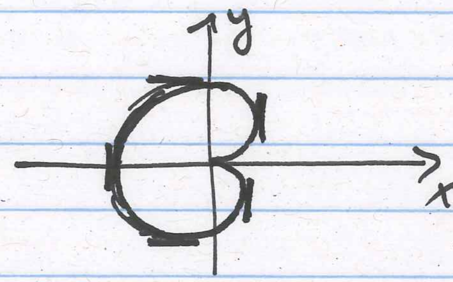
Ex Find points $-\pi \le \theta \le \pi$ at which cardioid $r = f(\theta) = 1-\cos\theta$ has a vertical or horizontal tangent line.

$f' = \sin\theta$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} =$$

$$= \frac{\sin^2\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}$$

$$= \frac{\overbrace{1-\cos^2\theta}^{(1-\cos\theta)(1+\cos\theta)} + (1-\cos\theta)\cos\theta}{\sin\theta\,(2\cos\theta - 1)} =$$



$$= \frac{(1-\cos\theta)(1+2\cos\theta)}{\sin\theta\,(2\cos\theta-1)}$$

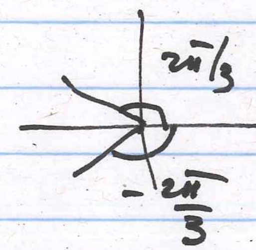
Tangent line is horizontal if $\frac{dy}{dx} = 0$

$$\Rightarrow (1-\cos\theta)(1+2\cos\theta) = 0$$

$1-\cos\theta = 0$ or $1+2\cos\theta = 0$ $\quad -\pi \le \theta \le \pi$

$\cos\theta = 1 \Rightarrow \theta = 0$

$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \pm\frac{2\pi}{3}$



Tangent line is vertical when

$\sin\theta = 0$ or $2\cos\theta = 1$

$\theta = 0, \pm\pi \qquad \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$

By L'Hopital's rule

$\frac{dy}{dx} = 0$ as $\theta = 0$

Exam #4 covers up to section 11.3.