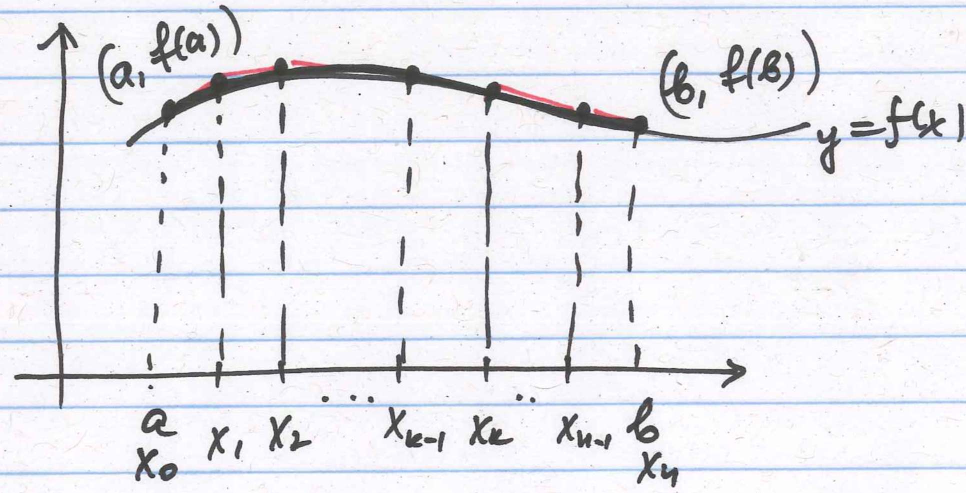
4/28/2014

6.5 Length of Curves

Arc length for $y=f(x)$

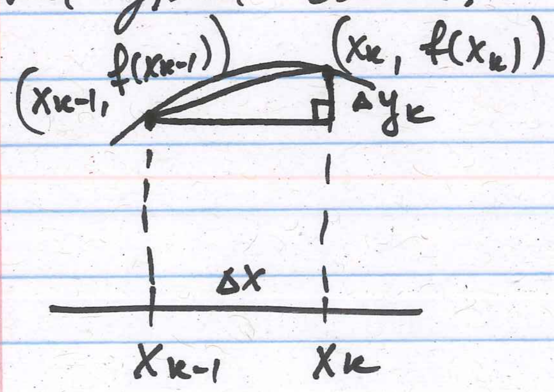


L = length of curve from $(a, f(a))$ to $(b, f(b))$

Subdivide $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$.

Partition: $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$

We connect points $(x_k, f(x_k))$ by straight lines to obtain a polygonal line whose length will be close to the length of the given curve.



$\Delta y_k = |f(x_k) - f(x_{k-1})|$

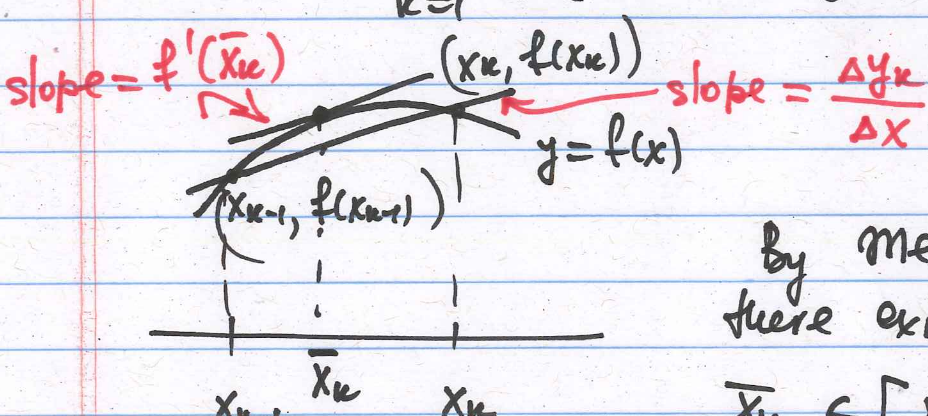
Length of segment connecting $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ is

$$\sqrt{(\Delta x)^2 + (\Delta y_k)^2} \qquad \text{or} \qquad \sqrt{\underbrace{(x_k - x_{k-1})^2}_{(\Delta x)^2} + \underbrace{\left(f(x_k) - f(x_{k-1})\right)^2}_{(\Delta y_k)^2}}$$

$$k = 1, 2, \ldots, n$$

Then

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_k)^2} \; = $$



By Mean Value Thm, there exists a point $\bar{x}_k \in [x_{k-1}, x_k]$:

$$f'(\bar{x}_k) = \frac{\Delta y_k}{\Delta x}$$

slope of tangent line at $x = \bar{x}_k$

slope of chord connecting end points

$$\Rightarrow \Delta y_k = f'(\bar{x}_k)\, \Delta x$$

$$= \sum_{k=1}^{n} \sqrt{(\Delta x)^2 + \left(f'(\bar{x}_k)\right)^2 (\Delta x)^2} =$$

$$= \sum_{k=1}^{n} \sqrt{1 + \left(f'(\bar{x}_k)\right)^2}\, \Delta x :$$ Riemann sum for function $\sqrt{1+\left(f'(x)\right)^2}$ and partition with mesh size Δx

Hence, $$L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1+\left(f'(\bar{x}_k)\right)^2}\, \Delta x =$$

$$= \int_a^b \sqrt{1+\left(f'(x)\right)^2}\, dx$$

Def Let $f(x)$ have a continuous 1^{st} derivative on $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\, dx$$

Ex Find length of curve $f(x) = x^{3/2}$ on $[0, 4]$.

$$f'(x) = \frac{3}{2} x^{\frac{1}{2}}, \quad a=0,\ b=4$$

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\, dx = \int_0^4 \sqrt{1+\left(\frac{3}{2} x^{\frac{1}{2}}\right)^2}\, dx$$

$$= \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx = \left| \begin{array}{c} u = 1+\frac{9}{4}x \\ du = \frac{9}{4}dx \\ x=0 \Rightarrow u=1 \\ x=4 \Rightarrow u=10 \end{array} \right| =$$

$$= \int_1^{10} \sqrt{u}\cdot\frac{4}{9}\, du = \frac{4}{9}\, \frac{u^{3/2}}{\frac{3}{2}} \Big|_1^{10} =$$

$$= \frac{4}{9}\cdot\frac{2}{3}\, u^{3/2} \Big|_1^{10} = \frac{8}{27}\left(10^{3/2}-1\right) \approx 9.1$$

Ex Find length of curve $f(x) = x^3 + \frac{1}{12x}$ on $[\frac{1}{2}, 2]$.

$$f'(x) = 3x^2 - \frac{1}{12x^2}$$

$$\left(f'(x)\right)^2 = \left(3x^2 - \frac{1}{12x^2}\right)^2 = 9x^4 - 2\cdot 3x^2\cdot\frac{1}{12x^2} +$$

$$+ \frac{1}{144x^4} = 9x^4 - \frac{1}{2} + \frac{1}{144x^4}$$

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\, dx =$$

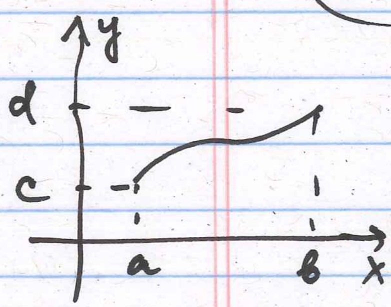
$$= \int_{\frac{1}{2}}^{2} \sqrt{1 + 9x^4 - \frac{1}{2} + \frac{1}{144x^4}}\, dx =$$

$$= \int_{\frac{1}{2}}^{2} \sqrt{9x^4 + \frac{1}{2} + \frac{1}{144x^4}}\, dx =$$

$$= \int_{\frac{1}{2}}^{2} \sqrt{\left(3x^2 + \frac{1}{12x^2}\right)^2}\, dx = \int_{\frac{1}{2}}^{2} \left(3x^2 + \frac{1}{12x^2}\right) dx =$$

$$= \left(x^3 - \frac{1}{12x}\right)\Big|_{x=\frac{1}{2}}^{2} = 2^3 - \frac{1}{12 \cdot 2} - \left(\left(\frac{1}{2}\right)^3 - \frac{1}{12 \cdot \frac{1}{2}}\right) =$$

$$= 8$$

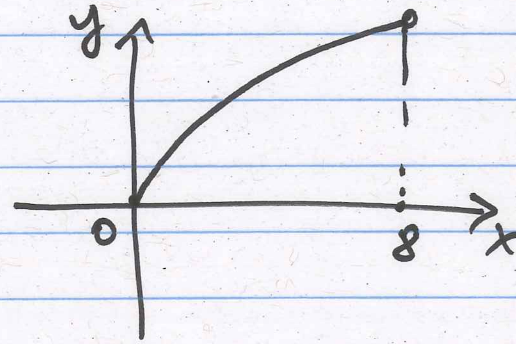


Arc length for $x=g(y)$

Let $x=g(y)$ have a continuous first derivative on $[c, d]$. The length of the curve from $(g(c), c)$ and $(g(d), d)$ is

$$L = \int_{c}^{d} \sqrt{1 + (g'(y))^2}\, dy$$

Ex Find length of curve $y = f(x) = x^{2/3}$ between $x=0$ and $x=8$.



$f'(x) = \frac{2}{3} x^{-1/3}$: not defined at $x=0$!

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

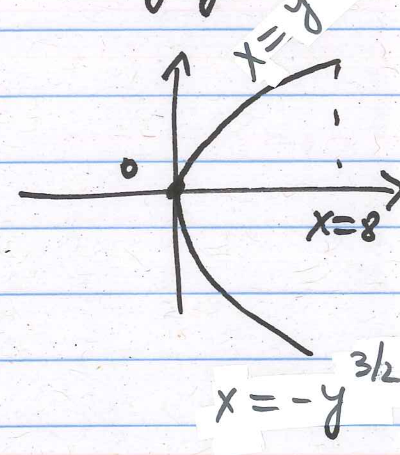
$$= \int_0^8 \sqrt{1+(f'(x))^2}\,dx$$

We cannot use formula for L wrt x since $f'(x)$ is not defined at $x=0$.

Let's see if we can use formulation wrt y.

$$L = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

$x = g(y)$



$y = x^{2/3} \Rightarrow y^3 = x^2 \geq 0$

$x = \pm y^{3/2}$

$x=0,\ x=8$

$x = 8 > 0 \Rightarrow$ we use positive branch $x = y^{3/2}$

$x = g(y) = y^{3/2} \Rightarrow g'(y) = \frac{3}{2} y^{\frac{1}{2}}$: defined at $y=0$!

$\Rightarrow L = \int_0^4 \sqrt{1+(g'(y))^2}\, dy = \int_0^4 \sqrt{1+\frac{9}{4}y}\, dy =$

$y = x^{\frac{2}{3}}$

$x=0 \Rightarrow y=0$

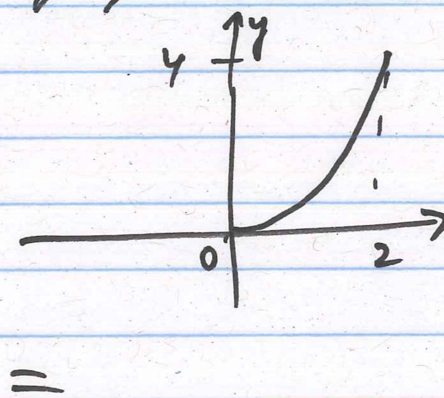
$x=8 \Rightarrow y = 8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^{3\cdot\frac{2}{3}}$

$= 2^2 = 4$

$\Rightarrow c=0,\ d=4$

see above ex #1 $= \frac{8}{27}(10^{3/2}-1) \approx 9.1$

ex Find length of segment of parabola $f(x)=x^2$ on $[0,2]$.



$f'(x) = 2x$

$L = \int_0^2 \sqrt{1+(f'(x))^2}\, dx =$

$= \int_0^2 \sqrt{1+(2x)^2}\, dx = \int_0^2 \sqrt{1+4x^2}\, dx \approx 4.647$

needs to be approximated numerically

$\int_a^b f(x)dx \approx \sum_{k=1}^{n} c_k f(x_k)$

4/29/2014

Calculus with Parametric Curves

Arc length

Recall, if we have a curve C given by eqn $y = f(x)$, $a \le x \le b$, then if $f'(x)$ is continuous on $[a, b]$, then length C of this curve is

(1) $$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now suppose that this curve C is described by parametric equations

$$x = g(t), \quad y = h(t), \quad \alpha \le t \le \beta$$

where

$$\frac{dx}{dt} = g'(t) > 0$$

which implies that C is traversed once from left to right as $t \nearrow$ from α to β and $g(\alpha) = a$, $g(\beta) = b$.

Recall, the slope $\frac{dy}{dx}$ of a tangent line is

(2) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{h'(t)}{g'(t)}$$

Substitute $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ into (1)

$$\Rightarrow \quad L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx =$$

$x = x(t)$

$$= \int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt =$$

$$= \int_\alpha^\beta \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, \frac{dx}{dt}\, dt \overset{\frac{dx}{dt}>0}{=}$$

$$\boxed{= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

since

$$\sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\, \frac{dx}{dt} =$$

$$\frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\left|\frac{dx}{dt}\right|} \cdot \frac{dx}{dt} = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\cancel{\frac{dx}{dt}}}\, \cancel{\frac{dx}{dt}}$$

$\left|\frac{dx}{dt}\right| = \frac{dx}{dt}$ since $\frac{dx}{dt} > 0$

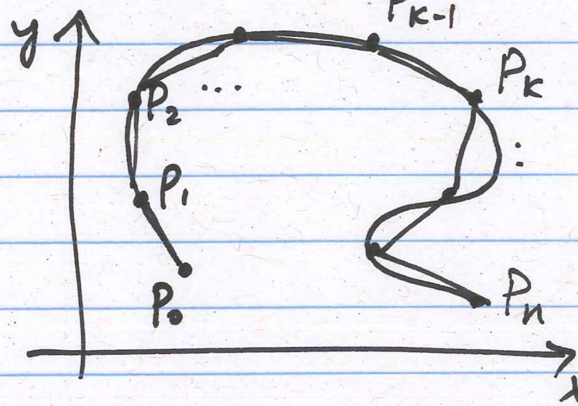
Hence,

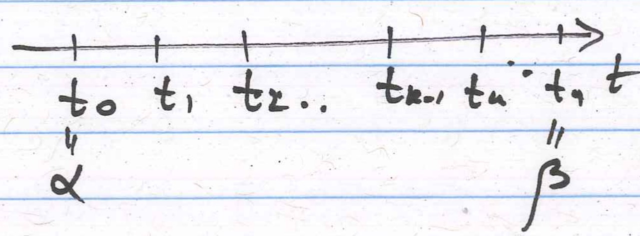
(3) $$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Even if C can't be expressed in $y=f(x)$, (3) is still valid. We can get by polygonal approximation.

$$x = g(t), \quad y = h(t), \quad \alpha \leq t \leq \beta$$

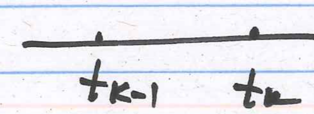
Subdivide $[\alpha, \beta]$ into n subintervals of length $\Delta t = \frac{\beta - \alpha}{n}$.





$x_k = g(t_k)$, $y_k = h(t_k)$: coordinates of points $P_k(x_k, y_k)$

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} |P_k P_{k-1}|$$

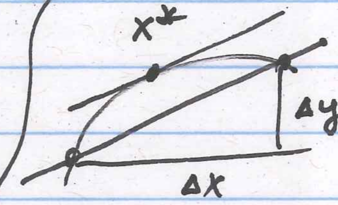


By Mean Value Thm applied to $g(t)$

exists

on $[t_{k-1}, t_k]$, $\exists\, t_k^* \in (t_{k-1}, t_k)$:

$$g(t_k) - g(t_{k-1}) = g'(t_k^*)(t_k - t_{k-1})$$



$$\frac{dy}{dx}(x^*) = \frac{\Delta y}{\Delta x}$$

$$\Downarrow$$

$$\Delta y = y'(x^*)\,\Delta x$$

Let $\Delta x_k = x_k - x_{k-1} = g(t_k) - g(t_{k-1})$

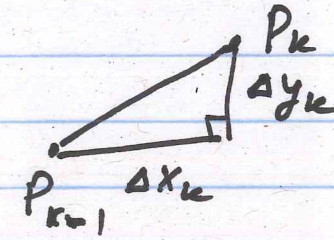
$\Delta y_k = y_k - y_{k-1} = h(t_k) - h(t_{k-1})$

$$\Rightarrow \quad \Delta x_k = g'(t_k^*)\,\Delta t$$

similarly, by mean value thm applied to $h(t)$ on $[t_{k-1}, t_k]$:

$$\Delta y_k = h'(t_k^{**})\,\Delta t, \qquad t_k^{**} \in (t_{k-1}, t_k)$$

Then



$$|P_{k-1} P_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} =$$

$$= \sqrt{\left(g'(t_k^*)\,\Delta t\right)^2 + \left(h'(t_k^{**})\,\Delta t\right)^2} =$$

$$= \sqrt{\left(g'(t_k^*)\right)^2 + \left(h'(t_k^{**})\right)^2}\;\Delta t$$

$$(4) \quad \Rightarrow \quad L = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\left(g'(t_k^*)\right)^2 + \left(h'(t_k^{**})\right)^2}\;\Delta t$$

resembles Riemann sum but $t_k^* \neq t_k^{**}$ in general

Nevertheless, if g' and h' are continuous $\Rightarrow$ limit (4) is the same as if $t_i^* = t_i^{**}$ and

$$L = \int_\alpha^\beta \sqrt{(g'(t))^2 + (h'(t))^2}\, dt$$

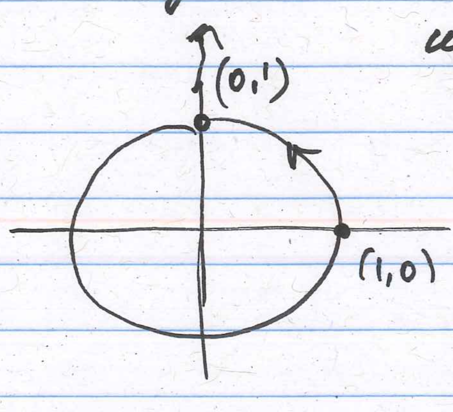
<u>Thm</u> If a curve C is described by parametric equations $x = g(t)$, $y = h(t)$, $\alpha \le t \le \beta$, where g' and h' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as $t \nearrow$ from α to β, then length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

<u>Ex</u> $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$\Rightarrow x^2 + y^2 = 1$: circle centered at $(0,0)$ with rad. 1 traversed once



$t = 0 \Rightarrow x = \cos 0 = 1$, $y = \sin 0 = 0$ $\quad (1,0)$

$t = \frac{\pi}{2} \Rightarrow x = 0$, $y = \sin\frac{\pi}{2} = 1$ $\quad (0,1)$

$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$$

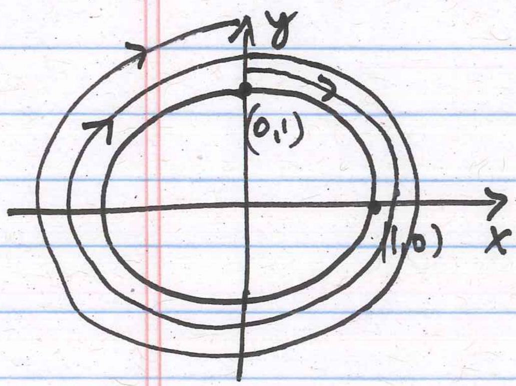
$$= \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt = \int_0^{2\pi} dt = 2\pi \checkmark$$

as expected

We need $2\pi r \Big|_{r=1} = 2\pi$ ✓

<u>Ex</u> $x = \sin 2t, \quad y = \cos 2t, \qquad 0 \le t \le 2\pi$

$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$: circle centered at $(0,0)$ with radius 1 but it makes two loops about $(0,0)$ as t varies from 0 to 2π



$t = 0 \Rightarrow x = \sin 0 = 0, \quad y = \cos 2\cdot 0 = 1$

$(0,1)$

$t = \pi/4 \Rightarrow x = 1, \quad y = 0 \Rightarrow (1,0)$

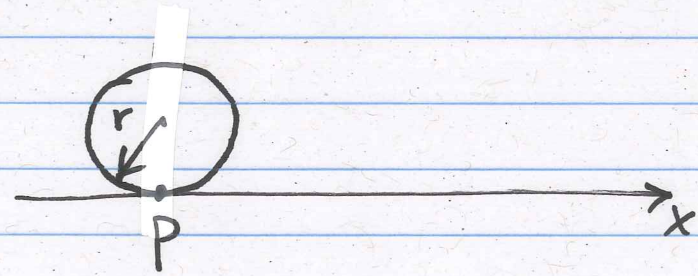
$\frac{dx}{dt} = 2\cos 2t, \quad \frac{dy}{dt} = -2\sin 2t$

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2}\, dt$$

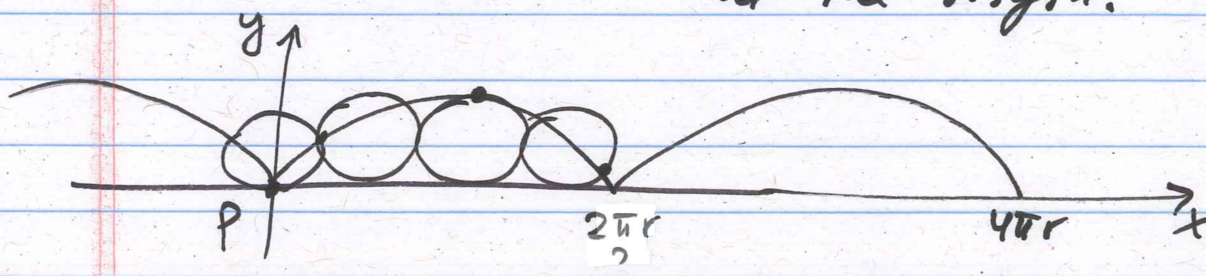
$$= 2\int_0^{2\pi} \sqrt{\cos^2 2t + \sin^2 2t}\, dt = 4\pi = 2 \cdot 2\pi r \Big|_{r=1}$$

twice the arclength

Ex



Take a circle of rad. r. Fix a point P on the circle and roll the circle along x-axis. As the circle rolls, the point P will describe a curve called cycloid. At the beginning P is at the origin.



Parametric equations of cycloid are

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta)$$

as $0 \le \theta \le 2\pi$, point P describes one arch.

$\theta = 0 \Rightarrow x = 0,\ y = 0$

at point ? : $y = 0 \Rightarrow \underset{\neq 0}{r}(1 - \cos\theta) = 0$

$1-\cos\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0, \pm 2\pi, \pm 4\pi, \ldots$

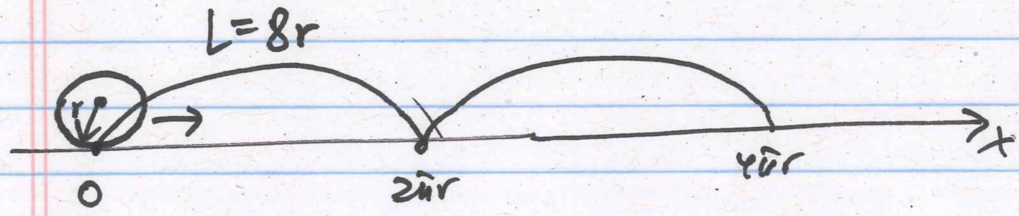
$\cos\theta = 1 \Rightarrow \sin\theta = 0, \quad x = r(\theta - \sin\theta)$

$\theta = 2\pi \Rightarrow \quad x = 2\pi r$

5/1/2014

Ex Find length of one arc of cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1-\cos\theta)$$



P One arch: $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta =$$

$$\frac{dx}{d\theta} = r(1-\cos\theta), \quad \frac{dy}{d\theta} = r\sin\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta =$$

$$= r\int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta =$$

$$= r\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}} = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

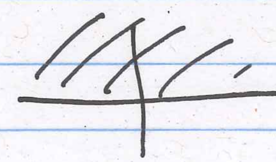
Recall $\sin^2 x = \dfrac{1-\cos 2x}{2}$ $\qquad \theta = 2x$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\Rightarrow 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\Rightarrow L = r\int_0^{2\pi}\sqrt{2\cdot 2\sin^2\frac{\theta}{2}}\,d\theta = r\int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta \;=$$

$$0 \le \theta \le 2\pi \;\Rightarrow\; 0 \le \frac{\theta}{2} \le \pi$$



$$\sin\frac{\theta}{2} \geq 0$$

$$\Rightarrow \left|\sin\frac{\theta}{2}\right| = \sin\frac{\theta}{2}$$

$$= 2r\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta = -4r\cos\frac{\theta}{2}\Big|_0^{2\pi} = -4r\Big(\underbrace{\cos\pi}_{=-1} - \underbrace{\cos 0}_{1}\Big) = 8r$$

$$\Rightarrow \boxed{L = 8r}$$

Arc length function

Recall, if f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$(1) \qquad L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

If a curve has eqⁿ $x = g(y)$, $c \le y \le d$ and $g'(y)$ is continuous on $[c, d]$, then

(2) $$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Ex Find length of arc of parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

$\overset{\displaystyle (0,0)}{\underset{c}{\shortparallel}}$ $\overset{\displaystyle (1,1)}{\underset{d}{\shortparallel}}$

We use (2).

$$x = y^2 \Rightarrow \frac{dx}{dy} = 2y$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy \;=$$

Trigonometric substitution: $y = \frac{1}{2}\tan\theta$

$dy = \frac{1}{2}\sec^2\theta\, d\theta$

$y = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

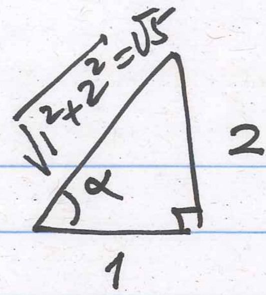
$y = 1 \Rightarrow \tan\theta = 2 \Rightarrow \theta = \tan^{-1} 2 \equiv \alpha$

$$1 + 4y^2 = 1 + \tan^2\theta = \sec^2\theta$$

$$= \int_0^\alpha \sqrt{\sec^2\theta}\, \frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta =$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^\alpha =$$

$\alpha = \tan^{-1} 2 \Rightarrow \tan\alpha = 2$



$\underbrace{\tan\alpha}_{\overset{\shortparallel}{2}}, \quad \sec\alpha = \frac{\sqrt{5}}{1} = \sqrt{5}$

$$\ominus \frac{1}{4}\left[\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right] -$$

$$-\frac{1}{4}\left[\cancelto{0}{\sec 0 \tan 0} + \cancelto{0}{\ln|\underbrace{\sec 0}_{1} + \tan 0|}\right] =$$

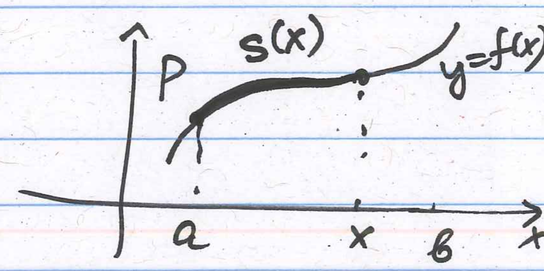
$$= \frac{1}{4}\left[\sqrt{5}\cdot 2 + \ln|\sqrt{5}+2|\right] = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}$$

Arc length function

$y = f(x), \qquad a \le x \le b$

Let $s(x)$ be the distance from some particular pt or initial pt $P_0(a, f(a))$ to another pt $Q(x, f(x))$ along a smooth curve. Then s is a function of x, called <u>arc length function</u>, and

$$s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\, dt$$



smooth function

$$\frac{ds}{dx} = \sqrt{1+[f'(x)]^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$\frac{ds}{dx}$: rate of change of distance s wrt x
and it is at least 1

$$\frac{ds}{dx}=0 \quad \text{if} \quad f'=0$$