4/02/2014

Exam # 3 Review

#11 S9.4

$$\sum_{k=1}^{\infty}\left[\frac{1}{3}\left(\frac{5}{6}\right)^k+\frac{3}{5}\left(\frac{7}{9}\right)^k\right]=$$

$$=\frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^k+\frac{3}{5}\sum_{k=1}^{\infty}\left(\frac{7}{9}\right)^k=$$

$$=\frac{1}{3}\cdot\frac{5}{6}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k-1}+\frac{3}{5}\cdot\frac{7}{9}\sum_{k=1}^{\infty}\left(\frac{7}{9}\right)^{k-1} =$$

Let $l=k-1$

$$= \frac{5}{18}\cdot\sum_{l=0}^{\infty}\left(\frac{5}{6}\right)^l+\frac{7}{15}\sum_{l=0}^{\infty}\left(\frac{7}{9}\right)^l =$$

$$\boxed{\sum_{k=0}^{\infty} r^k=\frac{1}{1-r},\quad |r|<1 \text{ for convergence : geometric series}}$$

$r_1=\frac{5}{6}<1,\quad r_2=\frac{7}{9}<1$

$$= \frac{5}{18}\cdot\frac{1}{1-\frac{5}{6}}+\frac{7}{15}\cdot\frac{1}{1-\frac{7}{9}}=\ldots$$

Ex Determine convergence or divergence. State which test you use.

#15 S9.4

$$\sum_{k=0}^{\infty}\underbrace{\frac{k}{2k+1}}_{a_n\to\frac{1}{2}\neq 0 \text{ as } k\to\infty} \Rightarrow \sum\frac{k}{2k+1} \text{ diverges by Divergence Test}$$

#23
S9.4

$$\sum_{k=2}^{\infty} \underbrace{\frac{1}{k \ln k}}_{a_k}$$

Use Integral Test.

$$a_k = f(k) \Rightarrow f(x) = \frac{1}{x \ln x}$$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x \ln x} = \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \right| = \int_{\ln 2}^{\infty} \frac{du}{u} =$$

$$= \ln u \Big|_{\ln 2}^{\infty} = \infty \Rightarrow \sum_{k=2}^{\infty} \frac{1}{k \ln k} \text{ diverges by Integral Test}$$

#46

$$\sum_{k=0}^{\infty} \frac{10}{k^2+9}$$

$$f(x) = \frac{10}{x^2+9} \qquad \int \frac{10\,dx}{x^2+9} = = \frac{10}{3}\arctan\frac{x}{3}\Big|_0^{\infty}$$

You can use integral test, comparison or limit comparison

Comparison Test:

$$\frac{10}{k^2+9} < \frac{10}{k^2} \qquad \sum \frac{10}{k^2} < \infty \text{ as } p\text{-series } p=2$$

$$\Rightarrow \sum \frac{10}{k^2+9} < \infty \text{ by Comparison Test}$$

Limit Comparison Test:

$$\frac{10}{k^2+9} \approx \frac{10}{k^2} \quad \text{for large } k$$

or $\lim_{k\to\infty} \dfrac{\frac{10}{k^2+9}}{\frac{10}{k^2}} = 1$: finite # $\neq 0$

since $\sum \frac{10}{k^2} < \infty \Rightarrow \sum \frac{10}{k^2+9} < \infty$ by Limit Compar. Test

#14 S9.5

$$\sum_{k=1}^{\infty} \frac{\overbrace{k!}^{a_k}}{k^k}$$

Use Ratio Test:

$$r = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} =$$

$(k+1)! = k!(k+1)$

$$= \lim_{k\to\infty} \frac{\cancel{k!}\cancel{(k+1)} \cdot k^k}{(k+1)^{k+\cancel{1}} \cdot \cancel{k!}} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = 1^\infty =$$

$$= \lim_{k\to\infty} e^{\ln\left(\frac{k}{k+1}\right)^k} = e^{\lim_{k\to\infty} k \ln \frac{k}{k+1}} = e^L$$

$$L = \lim_{k\to\infty} k \ln \frac{k}{k+1} = \infty \cdot 0 = \lim_{k\to\infty} \frac{\ln \frac{k}{k+1}}{\frac{1}{k}} = \frac{0}{0} \stackrel{H}{=}$$

use l'Hopital's Rule

$$\stackrel{H}{=} \lim_{k\to\infty} \frac{\cancel{\frac{k+1}{k}} \cdot \frac{(\cancel{k}+1) - \cancel{k}}{(k+1)^{\cancel{2}}}}{-\frac{1}{k^{\cancel{2}}}} = -\lim_{k\to\infty} \frac{k}{k+1} = -1$$

$$\Rightarrow r = e^{-1} = \frac{1}{e} \approx \frac{1}{2.718} < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{k!}{k^k} < 1 \text{ by Ratio Test}$$

$e = 2.718$

#21
S9.5

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

$$\rho = \lim \sqrt[k]{a_k} = = \lim_{k\to\infty}\left(\frac{k^2}{2^k}\right)^{\frac{1}{k}} = = \frac{1}{2} \lim_{k\to\infty} \left(k^2\right)^{\frac{1}{k}} \ (=)$$

$$\lim_{k\to\infty}\left(k^2\right)^{\frac{1}{k}} = = \lim_{k\to\infty} e^{\ln\left(k^2\right)^{\frac{1}{k}}} = e^{\lim_{k\to\infty} \frac{2}{k}\ln k} = e^0 = 1$$

$$(=)\ \frac{1}{2}\cdot 1 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{k^2}{2^k} < \infty \text{ by Root Test}$$

Ratio Test:

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^2}{2^{k+1}} \cdot \frac{2^k}{k^2} = \frac{1}{2} \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^2$$

$$= \frac{1}{2} < \infty \Rightarrow \sum \frac{k^2}{2^k} < \infty \text{ by Ratio Test}$$

#11
S9.6

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot a_k \qquad a_k = \frac{1}{k^3}$$

alternating series

$a_k = \frac{1}{k^3} \to 0$ monotonically

$$\therefore \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} < \infty \text{ by alt. series test}$$

#25 How many terms to keep to ensure that remainder is $< 10^{-4}$?

$$S = \sum_{k=0}^{\infty} (-1)^k a_k$$

$S = S_n + R_n$ or $|R_n| = |S - S_n| < a_{n+1}$

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \qquad a_k = \frac{1}{k}$$

$$|R_n| < a_{n+1} = \frac{1}{n+1} < 10^{-4}$$

$$n+1 > 10^4 \Rightarrow n > 10^4 - 1 = 9999$$

#14 Find Taylor polyn. for $f(x) = e^{-x}$, $a=0$

10.1?

Taylor polynomial for e^x is

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

since $f = f' = \dots = e^x$

Let $x \to -x \Rightarrow$

$$p_n(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots + \frac{(-x)^n}{n!}$$

Otherwise, compute derivatives and evaluate at $x = a$.

#53 Find max error (not unique).

10.1?

$$\sin x \approx x - \frac{x^3}{6} = P_3(x) \qquad x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

Taylor polynomial of degree $n = 3$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

ξ is between a and x

Here,

$$f(x) = \sin x$$

$$f^{(n+1)}(x) = \pm \sin x \quad \text{or} \quad \pm \cos x$$

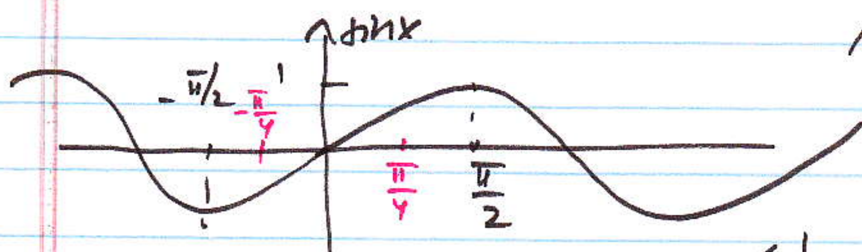
with $n = 3$

$$f^{(3+1)}(x) = \sin x$$

$$|f^{(4)}(x)| = |\sin x| \leq \overset{= M_1}{1} \quad \text{or even}$$

$$|\sin x| \leq \sin\frac{\pi}{4} = \overset{M_2}{\overset{\|}{\frac{\sqrt{2}}{2}}}$$

since $|x| \leq \frac{\pi}{4}$



$$|R_n(x)| \leq \frac{\overbrace{|f^{(n+1)}(\xi)|}^{\leq 1}}{(n+1)!}|x|^{n+1} \leq \frac{\overset{M_1}{1}}{(3+1)!}\left(\frac{\pi}{4}\right)^{3+1}$$

$a = 0$, $n = 3$

$$\text{or} \quad < \frac{\overset{= M_2}{\frac{\sqrt{2}}{2}}}{(3+1)!}\left(\frac{\pi}{4}\right)^{3+1} = \frac{\sqrt{2}}{2 \cdot 4!}\left(\frac{\pi}{4}\right)^4$$

etc.

Write expansion for

#33 $\frac{1}{(1-x)^2}$ using expansion for $\frac{1}{1-x}$

10.2

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k \text{: geom. series}$$

Note that

$$\frac{1}{(1-x)^2} = f'(x) = \sum_{k=0}^{\infty} k x^{k-1} = \sum_{k=1}^{\infty} k x^{k-1}$$

#21 Approximate $\sqrt{1.05}$ using $f(x) = \sqrt{1+x}$

10.1 and $p_2(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$

$$p_2(x) \approx \sqrt{1+x}$$

$$\sqrt{1.05} = \sqrt{1+0.05} \approx p_2(0.05) = \ldots$$

— Review abs. and conditional convergence.