MATH 275: Exam 1

FALL 2016

Prof. Lyudmyla Barannyk

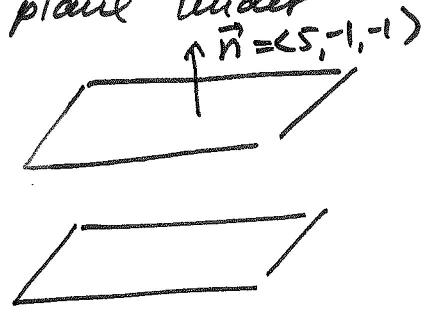
NAME: SOLUTIONS

For each problem clearly **BOX** your final answer. If you need additional space, continue your work on the back of the page or extra sheet at the end of the exam. **Calculators are not allowed.** Table of graphs of quadric surfaces is provided on page 7.

Problem	Points Possible	Points Earned
1	10	
2	15	
3	15	
4	30	
5	15	
6	15	
Total	100	

1. [**10 pts**] Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

plane $5x-y-z=6$ has normal vector $\vec{n}=\langle 5,-1,-1\rangle$ that is $\perp$ to the plane, and $\perp$ to the plane under question $\Rightarrow$ we can use $\vec{n}$ as a normal vector for a new plane



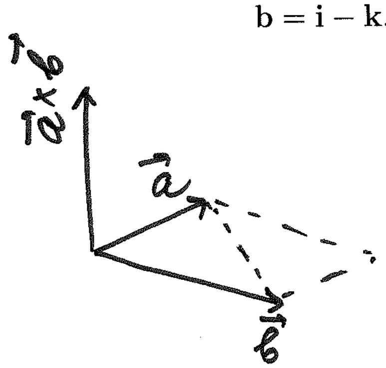
$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$(x_0,y_0,z_0)=(1,-1,-1)$, $a=5$, $b=-1$, $c=-1$

$5(x-1)-1(y-(-1))-1\cdot(z-(-1))=0$

$5(x-1)-(y+1)-(z+1)=0$ or $\boxed{5x-y-z-7=0}$

2. [**15 pts**] Find the area of the triangle determined by vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{k}$.



$$\vec{a}\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \hat{i}(-1)-\hat{j}(-1)+\hat{k}(-1)$$

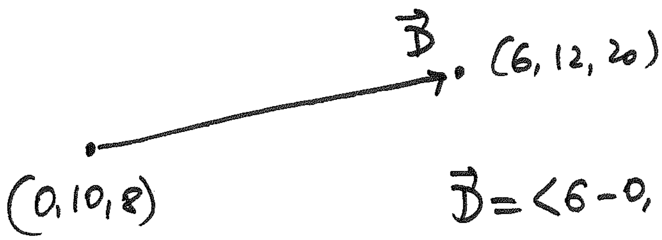
$$=-\hat{i}+\hat{j}-\hat{k}$$

$Area_{\square}=|\vec{a}\times\vec{b}|$
parallelogram determined by $\vec{a}$, $\vec{b}$

$$Area_{\Delta}=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\sqrt{(-1)^2+(-1)^2+(-1)^2}$$

$$=\frac{\sqrt{3}}{2}$$

3. [**15 pts**] Find the work done by a force $\mathbf{F} = 8\mathbf{i} - \mathbf{j} + 9\mathbf{k}$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.



$\vec{D} = \langle 6-0, 12-10, 20-8 \rangle = \langle 6, 2, 12 \rangle$
displacement vector

Work $W = \vec{F} \cdot \vec{D} = \langle 8, -1, 9 \rangle \cdot \langle 6, 2, 12 \rangle = 8 \cdot 6 - 2 + 9 \cdot 12 = 154 \text{ (J)}$

4. [**30 pts**]

   (a) [**15 pts**] Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

   $$x + y + z = 1, \quad x + 2y + 2z = 1$$

   (b) [**15 pts**] If the planes intersect, find parametric equations for the line of intersection of the planes.

(a)

Plane P_1 $x+y+z=1$ has normal vector $\vec{n}_1 = \langle 1, 1, 1 \rangle$

Plane P_2 $x+2y+2z=1$ has normal vector $\vec{n}_2 = \langle 1, 2, 2 \rangle$

$\vec{n}_1 \neq k\vec{n}_2 \Rightarrow$ planes are not parallel

$\vec{n}_1 \cdot \vec{n}_2 = \langle 1,1,1 \rangle \cdot \langle 1,2,2 \rangle = 1+2+2 = 5 \neq 0 \Rightarrow$ planes are not perpendicular $\Rightarrow$ they intersect at some angle $\theta \neq 0, \neq \frac{\pi}{2}$.

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{5}{\sqrt{1^2+1^2+1^2} \cdot \sqrt{1^2+2^2+2^2}} = \frac{5}{3\sqrt{3}}$$

$$\Rightarrow \quad \theta = \arccos\left(\frac{5}{3\sqrt{3}}\right)$$

#4 b Line of intersection L belongs to both planes $\Rightarrow L \perp \vec{n}_1$, $L \perp \vec{n}_2 \Rightarrow L \parallel \vec{n}_1 \times \vec{n}_2$. Let $\vec{v}$ be a direction vector of $L \Rightarrow \vec{v} \parallel \vec{n}_1 \times \vec{n}_2$. We can take

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \hat{i}\cdot 0 - \hat{j}\cdot 1 + \hat{k}\cdot 1 = \langle 0, -1, 1 \rangle$$

Need a point on L. This point should belong to both planes $\Rightarrow$ its coordinates (x, y, z) should satisfy both equations

$x + y + z = 1$, $\quad x + 2y + 2z = 1$: system of two eq^ns with three unknowns

$\Rightarrow$ ~~system~~ has ∞ many solutions. One variable is a free parameter.

Let $x = 0 \Rightarrow \begin{matrix} y + z = 1 \\ 2y + 2z = 1 \end{matrix} \Rightarrow 2(y+z) = 1 \Rightarrow y + z = \frac{1}{2}$

No solution

$\Rightarrow$ Let $y = 0 \Rightarrow \left.\begin{matrix} x + z = 1 \\ x + 2z = 1 \end{matrix}\right\} \Rightarrow z = 0 \Rightarrow x = 1$

$\Rightarrow$ pt $(1, 0, 0)$

Line: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, $t \in \mathbb{R}$

$(x_0, y_0, z_0) = (1, 0, 0)$, $\quad \langle a, b, c \rangle = \langle 0, -1, 1 \rangle$

$\Rightarrow x = 1 + 0\cdot t$, $y = 0 - 1\cdot t$, $z = 0 + 1\cdot t$, $t \in \mathbb{R}$

or $\boxed{x = 1,\ y = -t,\ z = t,\ t \in \mathbb{R}}$ parametric eq^ns of line of intersection

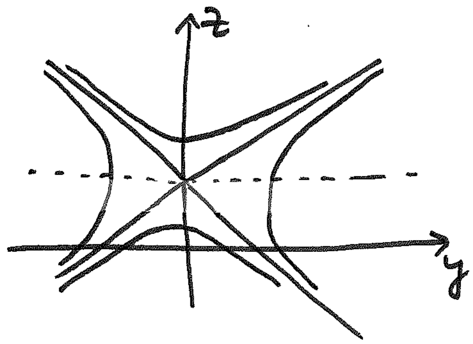
5. **[15 pts]** Reduce the equation to one of the standard forms, discuss traces, classify the surface, and sketch it.

$$x^2 - y^2 + z^2 - 4x - 2z = 0$$

$(x^2-4x+4)-4-y^2+(z^2-2z+1)-1=0$

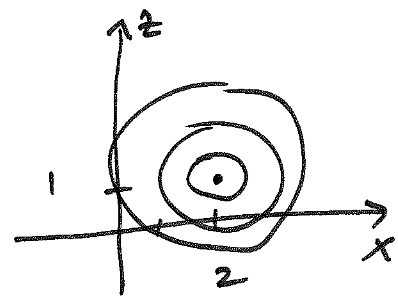
$(x-2)^2-y^2+(z-1)^2=5$ or $\frac{(x-2)^2}{5}-\frac{y^2}{5}+(z-1)^2=1$

Traces w/ $x=k$: $-y^2+(z-1)^2=5-(k-2)^2$: hyperbolas

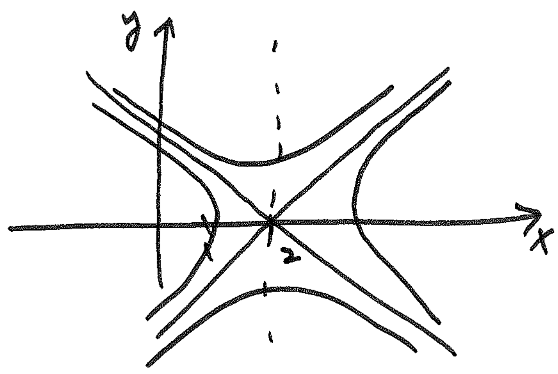


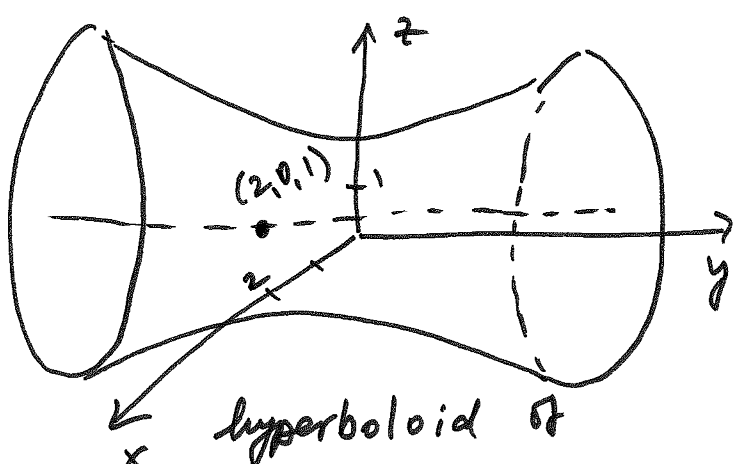
>0 for all k & y

Traces w/ $y=k$: $(x-2)^2+(z-1)^2=5+k^2$: circles



Traces w/ $z=k$:
$(x-2)^2-y^2=5-(k-1)^2$: hyperbolas

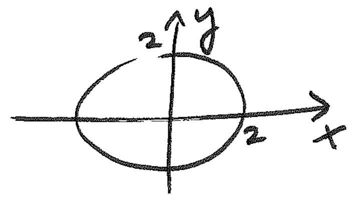




hyperboloid of one sheet

6. **[15 pts]** Find a vector function that represents the curve of intersection of the two surfaces: the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

Cylinder: $x^2+y^2=4$



$x = 2\cos t$
$y = 2\sin t$ $\quad 0 \le t \le 2\pi$

projection of cylinder on xy-plane is the circle $x^2+y^2=4$

surface $z=xy$

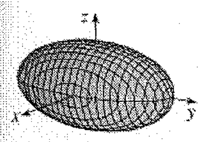
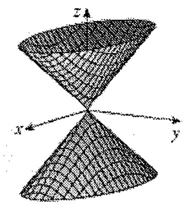
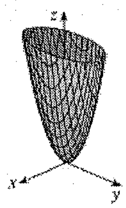
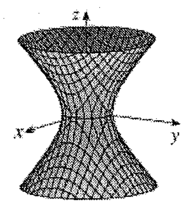
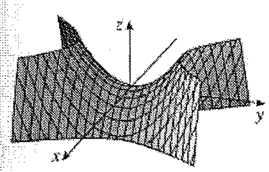
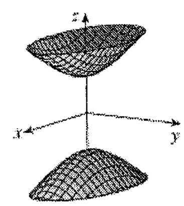
$\Rightarrow z = 2\cos t \cdot 2\sin t = 2\sin 2t$

$\Rightarrow x = 2\cos t$

$y = 2\sin t$ $\quad 0 \le t \le 2\pi$

$z = 2\sin 2t$

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.