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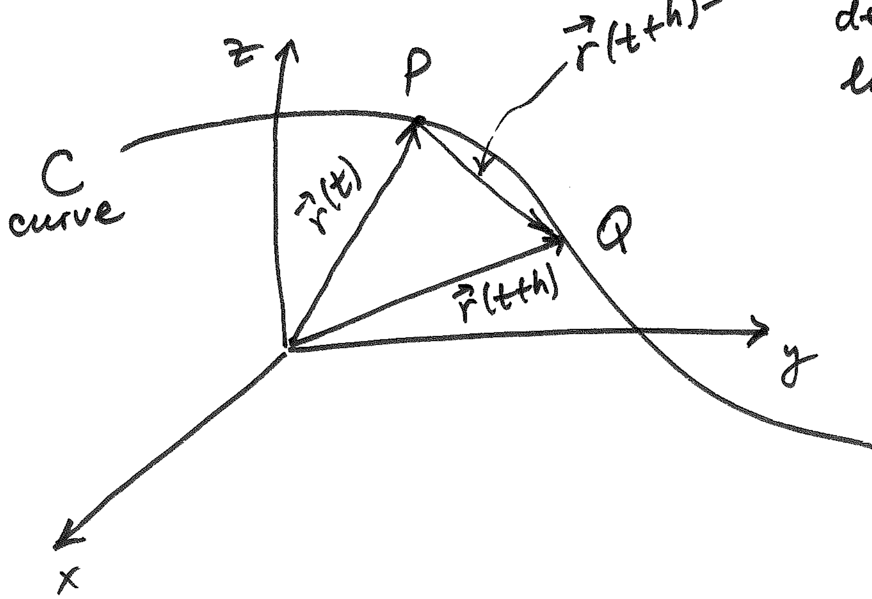
Lecture 10



$\vec{r}(t+h) - \vec{r}(t)$: defines direction of secant line PQ

$\dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$, $h>0$, has the same direction as $\vec{r}(t+h) - \vec{r}(t)$

as $h \to 0$ or $Q \to P$, secant line PQ approaches tangent line at P. Tangent line is parallel to $\vec{r}\,'(t)$.

$$\vec{T}(t) \overset{def}{=} \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad : \quad \underline{\text{unit tangent vector}}$$

Thm If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$, then
$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\hat{i} + g'(t)\,\hat{j} + h'(t)\,\hat{k}$
ie. we differentiate componentwise.

Ex

(a) Find derivative of $\vec{r}(t) = \langle \cos 3t, t, \sin 3t \rangle$

$\vec{r}\,'(t) = \langle -3\sin 3t, 1, 3\cos 3t \rangle$

(b) Find the unit tangent vector at the point when $t=0$.

$t=0 \Rightarrow x = \cos 3\cdot 0 = 1, \quad y = 0, \quad z = \sin 3\cdot 0 = 0 \Rightarrow$ pt $(1,0,0)$

$$\vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} =$$

$\vec{r}\,'(0) = \langle -3\sin 0, 1, 3\cos 0 \rangle = \langle 0, 1, 3 \rangle$

$|\vec{r}\,'(0)| = \sqrt{0^2+1^2+3^2} = \sqrt{10}$

$$= \frac{\langle 0, 1, 3 \rangle}{\sqrt{10}} = \left\langle 0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

(c) Find parametric eq^{ns} of the tangent line at $(1,0,0)$

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Parametric eq^{ns} of the line:

$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$: line through pt (x_0, y_0, z_0) in direction $\vec{v} = \langle a, b, c \rangle$

Here $(x_0, y_0, z_0) = (1, 0, 0)$.

For the tangent line at $(1,0,0)$, as the direction vector $\vec{v}$ we can use either $\vec{r}\,'(0)$ or $\vec{T}(0)$.

If $\vec{v} = \vec{r}\,'(0) = \langle 0, 1, 3 \rangle$, then

$x = 1 + 0 \cdot t, \quad y = 0 + 1 \cdot t, \quad z = 0 + 3 \cdot t, \quad t \in \mathbb{R}$

or $\boxed{x = 1, \quad y = t, \quad z = 3t, \quad t \in \mathbb{R}}$ tangent line at $(1,0,0)$

If $\vec{v} = \vec{T}(0) = \langle 0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$

$\boxed{x = 1, \quad y = \frac{1}{\sqrt{10}} t, \quad z = \frac{3}{\sqrt{10}} t, \quad t \in \mathbb{R}}$ another eq^{n} of tangent line at $(1,0,0)$

Differentiation Rules

$\vec{u}, \vec{v}$: differentiable vector functions

$f(t)$: differentiable scalar function

c: scalar

1. $\frac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$

2. $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\frac{d}{dt}\vec{u}(t)$

3. $\frac{d}{dt}\left[\underbrace{f(t)}_{\text{scalar}} \cdot \underbrace{\vec{u}(t)}_{\text{vector}}\right] = \frac{df}{dt} \cdot \vec{u}(t) + f(t) \cdot \frac{d\vec{u}(t)}{dt}$

4. $\frac{d}{dt}\left[\vec{u}(t) \underbrace{\cdot}_{\text{dot product}} \vec{v}(t)\right] = \frac{d\vec{u}}{dt} \cdot \vec{v}(t) + \vec{u}(t) \cdot \frac{d\vec{v}}{dt}$

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5. $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \frac{d\vec{u}}{dt}\times\vec{v}(t) + \vec{u}(t)\times\frac{d\vec{v}}{dt}$

6. $\frac{d}{dt}\left[\vec{u}(\underbrace{f(t)}_{y})\right] = \frac{d\vec{u}}{dy}\cdot\frac{dy}{dt}$ $\qquad y = f(t)$

$$\frac{dy}{dt} = \frac{df}{dt}$$

Ex Show that if $|\vec{r}(t)| = c = \text{const}$, then $\vec{r}\,'(t) \perp \vec{r}(t)$.

Solution

$|\vec{r}(t)| = c = \text{const} \Rightarrow |\vec{r}(t)|^2 = c^2$

but $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$

since $|\vec{r}(t)|^2 = c^2 = \text{const} \Rightarrow \frac{d}{dt}|\vec{r}(t)|^2 = 0$

$$\Rightarrow 0 = \frac{d}{dt}|\vec{r}(t)|^2 = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \frac{d\vec{r}(t)}{dt}\cdot\vec{r}(t) + \vec{r}(t)\cdot\frac{d\vec{r}(t)}{dt} =$$

$$= 2\,\frac{d\vec{r}(t)}{dt}\cdot\vec{r}(t) = 2\,\vec{r}\,'(t)\cdot\vec{r}(t)$$

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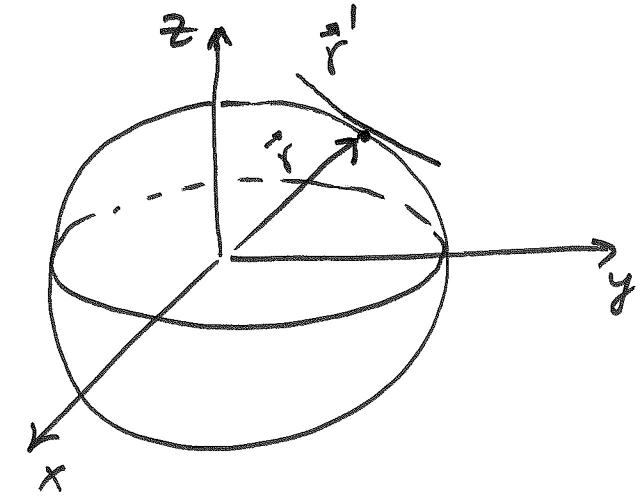
$\vec{r}\,'(t) \cdot \vec{r}(t) = 0 \Rightarrow \vec{r}\,'(t) \perp \vec{r}(t)$

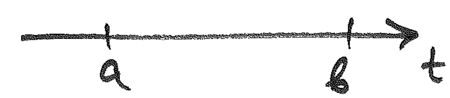


Geometrically: curve is on the sphere centered at origin: any position vector $\vec{r}(t)$ is $\perp$ to tangent vector $\vec{r}\,'(t)$.

INTEGRATION

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$

If $\vec{r}(t)$ is integrable, then



$$\int_a^b \vec{r}(t)\,dt = \int_a^b \left[f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} \right] dt$$

definite integral

$$\boxed{\int_a^b \vec{r}(t)\,dt = \int_a^b f(t)\,dt \cdot \hat{i} + \int_a^b g(t)\,dt \cdot \hat{j} + \int_a^b h(t)\,dt \cdot \hat{k}}$$

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Fundamental Theorem of Calculus

Let $\vec{r}(t)$ be a continuous ~~vector~~ function. Then

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_{t=a}^{t=b} = \vec{R}(b) - \vec{R}(a)$$

where $R(t)$ is an antiderivative of $\vec{r}(t)$, i.e. $\vec{R}'(t) = \vec{r}(t)$.

<u>Notation</u>: $\int \vec{r}(t)\,dt$: indefinite integral.

Ex For $\vec{r}(t) = \dfrac{4}{1+t^2}\,j + \dfrac{2t}{1+t^2}\,k$, evaluate integrals

$\int_0^1 \vec{r}(t)\,dt$ and $\int \vec{r}(t)\,dt$.

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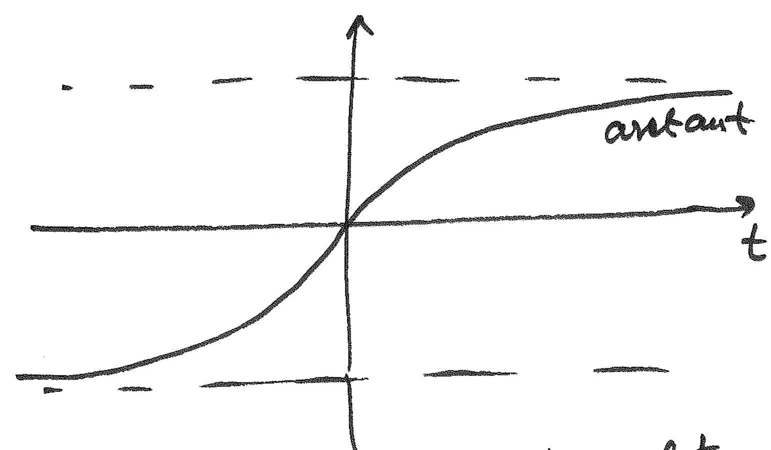
Solution:

$$\int_0^1 \vec{r}(t)\,dt = \int_0^1 \left(\frac{4}{1+t^2}\vec{j} + \frac{2t}{1+t^2}\vec{k} \right) dt =$$

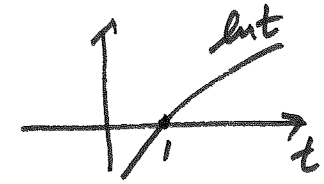
$$= \int_0^1 \frac{4}{1+t^2}\,dt \cdot \vec{j} + \int_0^1 \frac{2t}{1+t^2}\,dt \cdot \vec{k} \;\boxed{=}$$

$$\int_0^1 \frac{4}{1+t^2}\,dt = 4\arctan t \Big|_{t=0}^{1} = 4(\arctan 1 - \cancel{\arctan 0}^{\,0}) =$$

$$= 4 \cdot \frac{\pi}{4} = \boxed{\pi}$$



$$\int_0^1 \frac{2t}{1+t^2}\,dt = \left| \begin{array}{l} u = 1+t^2 \\ du = 2t\,dt \\ t=0 \Rightarrow u=1 \\ t=1 \Rightarrow u=2 \end{array} \right| =$$



$$= \int_1^2 \frac{du}{u} = \ln u \Big|_1^2 = \ln 2 - \cancel{\ln 1}^{\,0} = \boxed{\ln 2}$$

$\boxed{=}$ $\pi \cdot \vec{j} + \ln 2 \cdot \vec{k} = \langle 0, \pi, \ln 2 \rangle$

$$\int \vec{r}(t)\,dt = \int \left(\frac{4}{1+t^2} \vec{j} + \frac{2t}{1+t^2} \vec{k} \right) dt =$$

$$= \int \frac{4}{1+t^2}\,dt \cdot \vec{j} + \int \frac{2t}{1+t^2}\,dt \cdot \vec{k} + \underbrace{\vec{C}}_{\text{const vector of integration}} =$$

$\vec{C} = \langle C_1, C_2, C_3 \rangle$

$= 4 \arctan t \cdot \vec{j} + \ln(1+t^2) \cdot \vec{k} + \vec{C} =$

$= \langle C_1,\ 4 \arctan t + C_2,\ \ln(1+t^2) + C_3 \rangle$