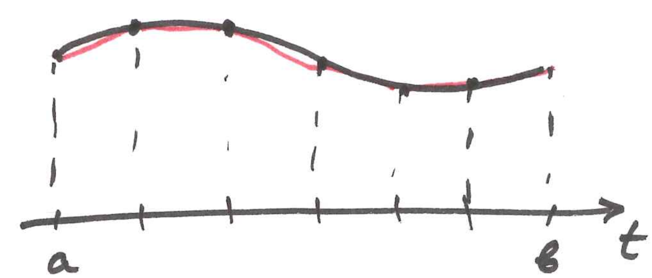
Lecture 11

13.3 Arc Length and Curvature

Plane curve: $x = f(t)$, $y = g(t)$, $a \le t \le b$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



length of curve = limit of length of inscribed polygon

provided f', g' are continuous, and the curve is traversed only once as $t \nearrow$ from a to b.

Space curve: $x = f(t)$, $y = g(t)$, $z = h(t)$, $a \le t \le b$

$$\boxed{L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

length of space curve

provided that f', g', h' are continuous, and curve is traversed once as $t \nearrow$ from a to b.

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ $\qquad$ $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$$\therefore \quad \boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt}$$

length of a curve: plane or space curve

<u>Ex</u> Find the length of curve

$\vec{r}(t) = t^2\vec{i} + 2t\vec{j} + \ln t\,\vec{k}, \qquad 1 \le t \le e$

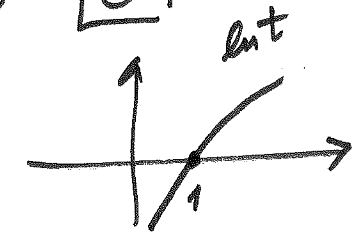
$\vec{r}\,'(t) = 2t\vec{i} + 2\vec{j} + \frac{1}{t}\vec{k}$

<u>Solution</u>

$$L = \int_1^e \sqrt{(2t)^2 + 2^2 + \left(\frac{1}{t}\right)^2}\, dt = \int_1^e \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt =$$

$$= \int_1^e \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \int_1^e \sqrt{\frac{(2t^2+1)^2}{t^2}}\, dt = \int_1^e \frac{2t^2+1}{t}\, dt =$$

$$= \int_1^e \left(2t + \frac{1}{t}\right) dt = \left(t^2 + \ln t\right)\Big|_1^e = e^2 + \underbrace{\ln e}_{=1} - (1^2 + \cancel{\ln 1}^{\,0}) = \boxed{e^2}$$



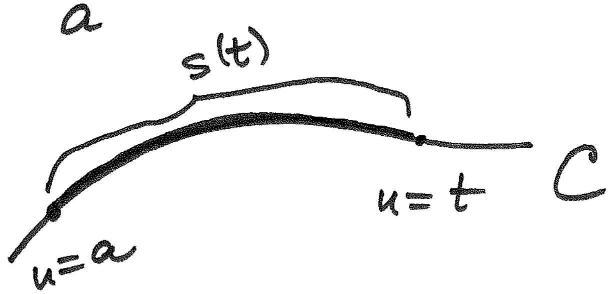
<u>Note</u> Curve can be parametrized using different parametrizations, but the length of the curve will be the same.

Consider

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$: curve C, piecewise smooth

Define <u>arc length function</u> as

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du = \int_a^t \sqrt{\left(\frac{df}{du}\right)^2 + \left(\frac{dg}{du}\right)^2 + \left(\frac{dh}{du}\right)^2}\,du$$



$$\boxed{\frac{ds}{dt} = |\vec{r}\,'(t)|}$$

Recall $\dfrac{d}{dt}\displaystyle\int_a^t f(u)\,du = f(t)$

It is convenient to parametrize a curve using arc length s as a parameter since it does not depend on coordinate system.

Ex Reparametrize the curve with respect to arc length measured from the point where $t=0$ in the direction of increasing t.

$a=0$

$$\vec{r}(t) = 3\sin t \cdot \vec{i} + 4t \cdot \vec{j} + 3\cos t \cdot \vec{k}$$

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

$$\vec{r}\,'(t) = 3\cos t \cdot \vec{i} + 4\vec{j} - 3\sin t \cdot \vec{k}$$

$$|\vec{r}\,'(t)| = \sqrt{(3\cos t)^2 + 4^2 + (-3\sin t)^2} = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$$

$$\Rightarrow |\vec{r}\,'(u)| = 5$$

$$\therefore \quad s(t) = \int_0^t 5\,du = 5u\Big|_{u=0}^{u=t} = 5t \Rightarrow \underbrace{\boxed{s = 5t}}_{\text{arc length}} \Rightarrow \boxed{t = \frac{s}{5}}$$

$\Rightarrow \boxed{\vec{r}(s) = 3\sin\frac{s}{5}\,\hat{i} + 4\cdot\frac{s}{5}\,\hat{j} + 3\cos\frac{s}{5}\cdot\hat{k}}$

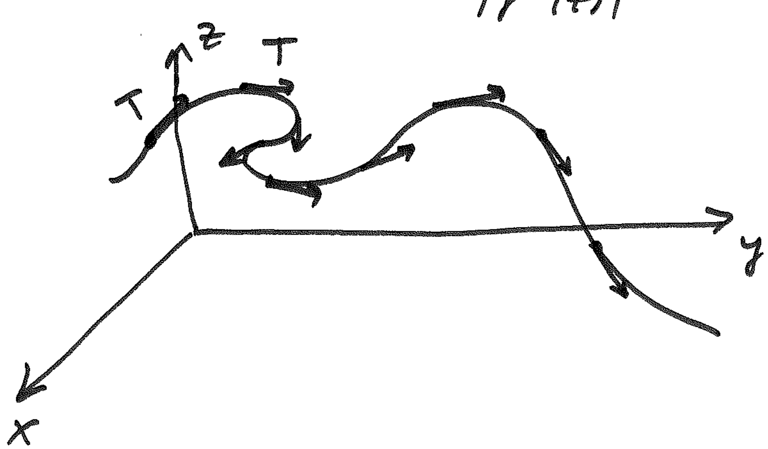
<u>Note</u>: parametrization of a curve wrt arc length is called NATURAL PARAMETRIZATION.

<u>Def</u> The curve C given by $\vec{r}(t)$ is <u>smooth</u> if $\vec{r}(t)$ is continuous and $\vec{r}\,' \not\equiv 0$ (except at endpoints)

Let C be a smooth curve.

Recall

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad : \text{unit tangent vector}$$



<u>Note</u>: $\vec{T}$ changes slowly when curve is $\approx$ straight line, and changes quickly when curve bends or twists quickly.

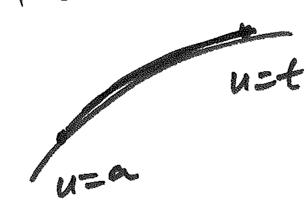
<u>**Def**</u> <u>Curvature</u> is measure of how quickly $\vec{T}(t)$ changes its direction. <u>Curvature</u> is the rate of change of $\vec{T}(t)$ wrt arclength (so that curvature does not depend on parametrization).

$$K = \left| \frac{d\vec{T}}{ds} \right| \quad : \quad \underline{\text{curvature}}$$

$\varkappa$

<u>**Note**</u> It is convenient to use parametrization wrt t instead of s.

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, dt$$



$$\vec{T} = \vec{T}(t)$$

$$\frac{d\vec{T}}{dt} \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{d\vec{T}}{ds} \cdot \underbrace{\frac{ds}{dt}}_{\substack{\shortparallel \\ |\vec{r}\,'(t)|}} \quad \Rightarrow$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

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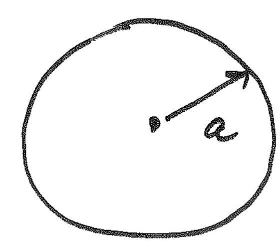
$$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{|\vec{r}\,'(t)|}$$

$$\therefore \quad \left|\frac{d\vec{T}}{ds}\right| = \kappa = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}\,'(t)|}$$

$$\Rightarrow \boxed{\kappa = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}\,'(t)|}} \quad : \quad \underline{\text{curvature}}$$

Ex Show that curvature of a circle of radius a is $\frac{1}{a}$.



$\vec{r}(t) = a\cos t \cdot \hat{\imath} + a\sin t \cdot \hat{\jmath}$

$\vec{r}\,'(t) = -a\sin t \cdot \hat{\imath} + a\cos t \cdot \hat{\jmath}$

$|\vec{r}\,'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a$

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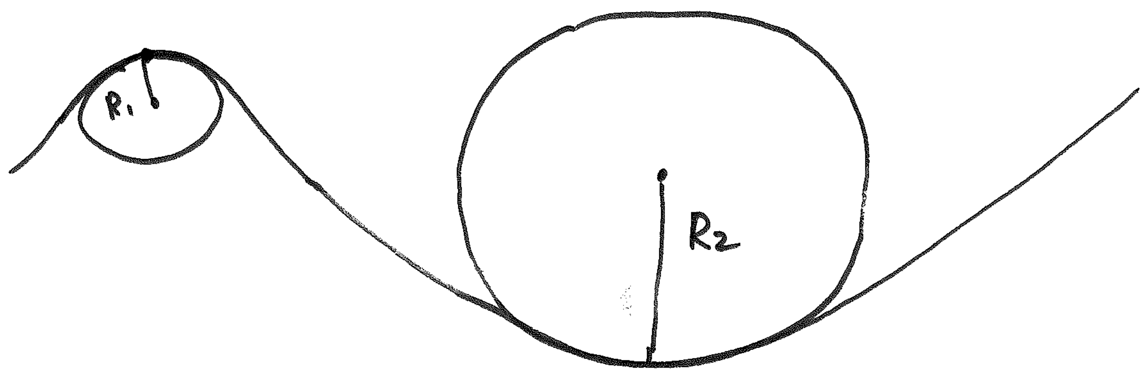
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{a}\left(-a\sin t\cdot\vec{i} + a\cos t\cdot\vec{j}\right) = -\sin t\cdot\vec{i} + \cos t\cdot\vec{j}$$

$$\frac{d\vec{T}}{dt} = -\cos t\cdot\vec{i} - \sin t\cdot\vec{j} \qquad \left|\frac{d\vec{T}}{dt}\right| = 1$$

$$\Rightarrow \quad K = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}\,'(t)|} = \frac{1}{a} \quad \Rightarrow \quad \boxed{K = \frac{1}{a}}$$



$R_1 < R_2$ but

$K_1 = \frac{1}{R_1}$ $\quad K_2 = \frac{1}{R_2}$

$K_1 > K_2$

<u>Note</u> Small circles have larger curvature, and large circle have smaller curvature.