Lecture 12

<u>Note</u> Curvature of a straight line is 0 since $\vec{T} = \overrightarrow{\text{const}}$ or you can think of a straight line as of a large circle whose radius $\to \infty$

<u>Thm</u> The curvature of the curve given by $\vec{r}(t)$ is

$$\boxed{K(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

<u>Ex</u> Plane curve $y = f(x)$.

Parametrization : $x = x,\ y = f(x)$, x is a parameter.

$\Rightarrow \vec{r}(x) = x \cdot \hat{\imath} + f(x) \cdot \hat{\jmath}$; $\vec{r}\,'(x) = 1 \cdot \hat{\imath} + f'(x) \cdot \hat{\jmath}$; $\vec{r}\,''(x) = f''(x)\,\hat{\jmath}$

$$\vec{r}\,'(x) \times \vec{r}\,''(x) = (\hat{\imath} + f'(x)\hat{\jmath}) \times f''(x)\hat{\jmath} = f''(x) \cdot \underbrace{(\hat{\imath} \times \hat{\jmath})}_{=\hat{k}} + f'(x) \cdot f''(x) \cdot \underbrace{(\hat{\jmath} \times \hat{\jmath})}_{=\vec{0}} =$$

$$= f''(x) \cdot \hat{k}$$

$$|\vec{r}\,'(x) \times \vec{r}\,''(x)| = |f''(x)|$$

$$|\vec{r}\,'(x)| = \sqrt{1 + (f'(x))^2}$$

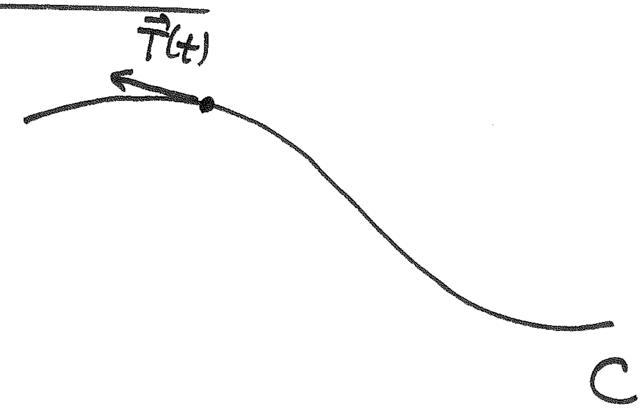
$$\therefore \quad K(x) = \frac{|f''(x)|}{\left(\sqrt{1+(f'(x))^2}\right)^3} = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$$

$$\therefore \quad \boxed{K(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}}$$

curvature of a plane curve

$y = f(x)$

The Normal and Binormal Vectors



Consider a curve C.

$\vec{T}$: unit tangent vector.

There are ∞ many vectors that are orthogonal to $\vec{T}$.

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Recall, we showed earlier, that if $|\vec{r}(t)| = \text{const} \Rightarrow \vec{r}(t) \perp \vec{r}\,'(t)$.

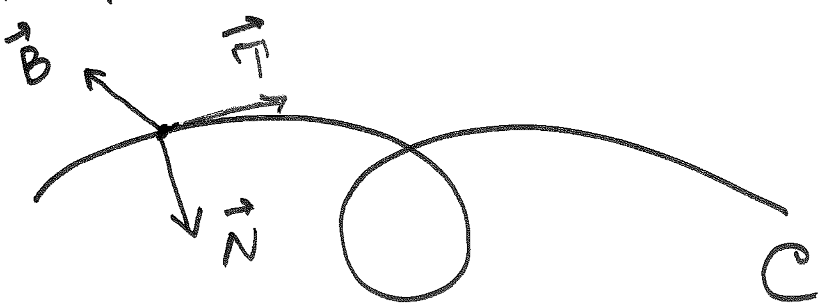
Note that $|\vec{T}| = 1 = \text{const} \Rightarrow \vec{T}\,' \perp \vec{T}$ or $\vec{T}\,' \cdot \vec{T} = 0$

Define

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} :$$ <u>principal normal vector</u>

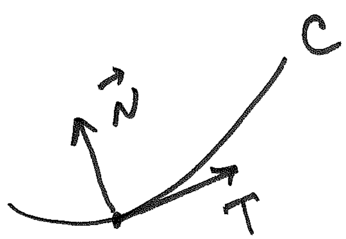
$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) :$$ <u>binormal vector</u>

<u>Note</u>: $\vec{B} \perp \vec{T}$, $\vec{B} \perp \vec{N}$ and $\vec{T}, \vec{N}, \vec{B}$ form a right-hand oriented triple.

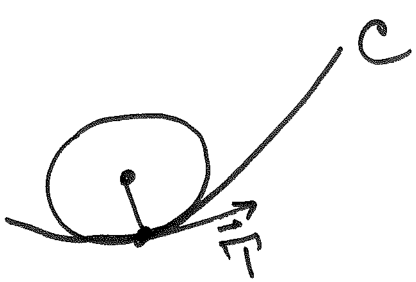


Def Plane determined by $\vec{N}$ and $\vec{B}$ is called <u>normal plane.</u> Note that $\vec{T}$ is $\perp$ to normal plane.

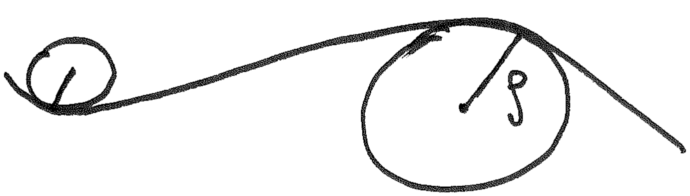
Def Plane determined by $\vec{T}$ and $\vec{N}$ is called <u>osculating plane.</u> $\vec{B} \perp$ osculating plane.



<u>Note</u>: $\vec{N}$ points to a concave side of curve C



The circle that lies in the osculating plane, has the same curvature as the curve, has the same tangent vector $\vec{T}(t)$, lies on concave side of C (where $\vec{N}$ points to) is called an <u>osculating circle.</u>



$$\rho = \frac{1}{K}$$

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Ex Find the unit normal and binormal vectors for the circular helix:

$$\vec{r}(t) = \cos t \cdot \hat{i} + \sin t \cdot \hat{j} + t \cdot \hat{k}$$

Solution

$$\vec{r}\,'(t) = -\sin t \cdot \hat{i} + \cos t \cdot \hat{j} + \hat{k} \; ; \quad |\vec{r}\,'(t)| = \sqrt{2}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{2}}\left(-\sin t \cdot \hat{i} + \cos t \cdot \hat{j} + \hat{k}\right)$$

unit tangent vector

$$\vec{T}\,'(t) = \frac{1}{\sqrt{2}}\left(-\cos t \cdot \hat{i} - \sin t \cdot \hat{j}\right) ; \qquad \vec{T}\,'(t) = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = -\cos t \cdot \hat{i} - \sin t \cdot \hat{j}$$: principal normal vector

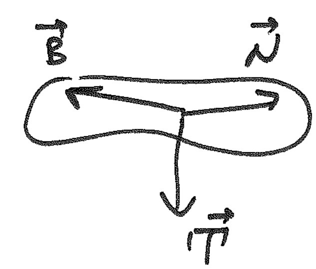
$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{\sin t}{\sqrt{2}}\hat{i} - \frac{\cos t}{\sqrt{2}}\hat{j} + \frac{\hat{k}}{\sqrt{2}}$$

binormal vector

Ex Find equations of normal plane and osculating plane of the helix from previous example at $P(0, 1, \frac{\pi}{2})$.

<u>Solution</u>

Normal plane is determined by $\vec{N}$ and $\vec{B}$ $\Rightarrow \vec{T} \perp$ normal plane



$$\vec{T}(t) = \frac{1}{\sqrt{2}}(-\sin t \cdot \hat{\imath} + \cos t \cdot \hat{\jmath} + \vec{k})$$

or we can use $\vec{r}\,'(t) = -\sin t \cdot \hat{\imath} + \cos t \cdot \hat{\jmath} + \vec{k}$

for simplicity since $\vec{T} \parallel \vec{r}\,'$.

Helix is parametrized by $\vec{r}(t) = \cos t \cdot \hat{\imath} + \sin t \cdot \hat{\jmath} + t \cdot \vec{k}$

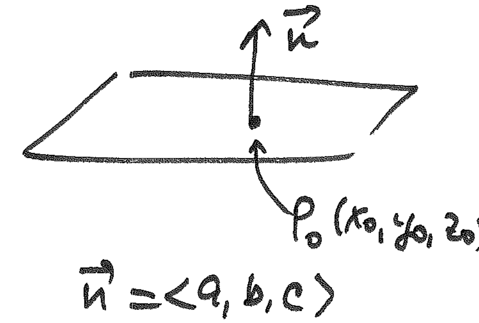
at P: $P(0, 1, \frac{\pi}{2})$

$$\left.\begin{array}{l} \cos t = 0 \\ \sin t = 1 \\ t = \frac{\pi}{2} \end{array}\right\} \Rightarrow \boxed{t = \frac{\pi}{2}} \text{ at } P$$

$$\Rightarrow \vec{r}\,'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} \cdot \hat{\imath} + \cos\frac{\pi}{2} \cdot \hat{\jmath} + \hat{k} = -\hat{\imath} + \hat{k} = \langle -1, 0, 1 \rangle$$

Plane:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$



Here $\langle a,b,c\rangle = \vec{r}\,'(\frac{\pi}{2}) = \langle -1,0,1\rangle$

$(x_0,y_0,z_0) = (0,1,\frac{\pi}{2})$

$$\therefore\quad -1\,(x-0)+\underbrace{0\cdot(y-1)}_{0}+1\cdot(z-\tfrac{\pi}{2})=0$$

$\boxed{x = z-\frac{\pi}{2}}$ normal plane at $P(0,1,\frac{\pi}{2})$

Osculating plane is determined by $\vec{N}$ and $\vec{T}$ $\Rightarrow$ $\vec{B}\perp$ osculating plane. We found

$$\vec{B}=\frac{1}{\sqrt{2}}(\sin t\,\vec{i}-\cos t\,\vec{j}+\vec{k})$$

or we can use $\sqrt{2}\vec{B} = \sin t\cdot\vec{i}-\cos t\,\vec{j}+\vec{k} \parallel \vec{B}$

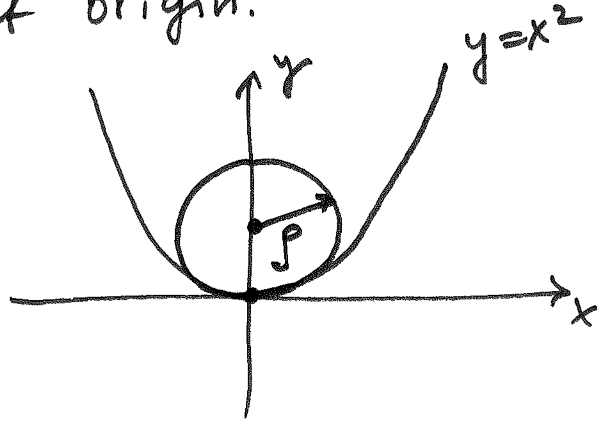
$$\sqrt{2}\vec{B}\Big|_{t=\frac{\pi}{2}} = \langle \sin\tfrac{\pi}{2}, -\cos\tfrac{\pi}{2}, 1\rangle = \langle 1, 0, 1\rangle$$

$P(0, 1, \frac{\pi}{2})$ $\quad \langle a, b, c\rangle = \langle 1, 0, 1\rangle$

$$\Rightarrow 1\cdot(x-0) + \underbrace{0\cdot(y-1)}_{0} + 1\cdot(z-\tfrac{\pi}{2}) = 0$$

$\boxed{x + z = \frac{\pi}{2}}$ oscullating plane at $P(0, 1, \frac{\pi}{2})$

Ex Find and graph oscullating circle of parabola $y = x^2$, at origin.
plane curve



$\rho = \frac{1}{K}$: radius of oscullating circle

K : curvature

$$K = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

$K = K(x)$

curvature of plane curve $y = f(x)$

Here $f(x) = x^2, \quad f'(x) = 2x, \quad f''(x) = 2$

$$K(x) = \frac{|2|}{\left[1 + (2x)^2\right]^{3/2}} = \frac{2}{\left[1 + 4x^2\right]^{3/2}} ; \qquad K(0) = 2$$

$\Rightarrow \rho = \frac{1}{K(0)} = \frac{1}{2} \Rightarrow$ center of oscullating circle is at $(0, \frac{1}{2})$
by symmetry

$$(x-0)^2 + (y - \tfrac{1}{2})^2 = \left(\tfrac{1}{2}\right)^2$$

or $\boxed{x^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}}$ oscullating circle at $(0,0)$