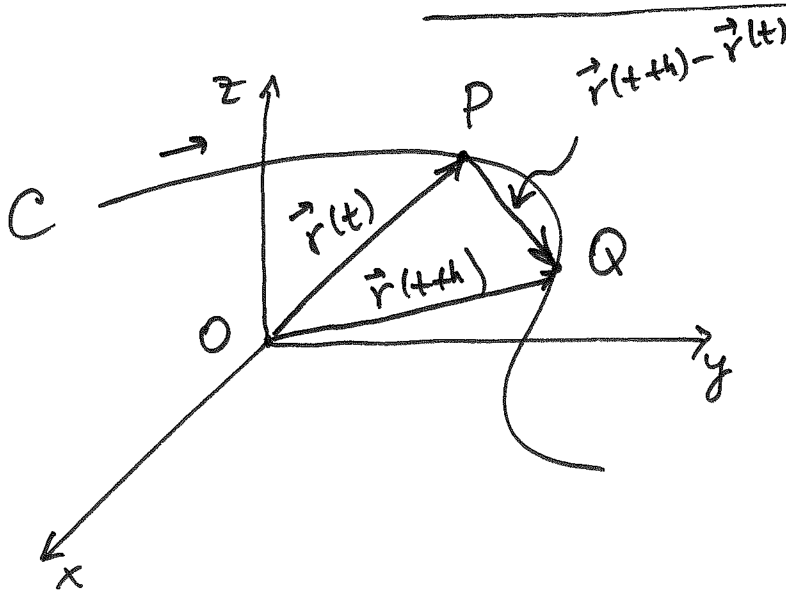
Lecture 14

13.4 Motion in Space : Velocity & Acceleration



Particle moves in space along curve $\vec{r}(t)$.

$\dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$ shows direction of motion along C.

Its magnitude measures the size of displacement per unit time.

$\dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$: average velocity of a particle over time interval h

$$\vec{v}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$: instantaneous velocity at pt P

Note : $\vec{v}(t) = \vec{r}\,'(t)$

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$: acceleration

$|\vec{v}(t)| = v$: speed of a particle

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt}$$

rate of change of distance wrt t

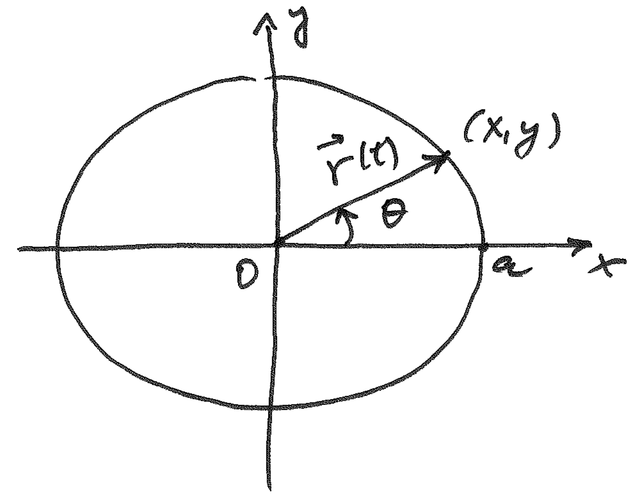
$$\vec{a}(t) = \vec{v}\,'(t) \Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \vec{v}(t_0) + \int_{t_0}^{t} \vec{a}(u)\,du$$

$$\vec{v}(t) = \vec{r}\,'(t) \Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt = \vec{r}(t_0) + \int_{t_0}^{t} \vec{v}(u)\,du$$

Consider a particle of mass m moving under effect of force $\vec{F}(t)$. Motion obeys 2nd Newton's law:

$$\vec{F} = m\vec{a}$$

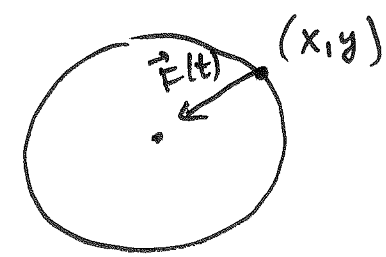
Ex Object of mass m moves in a circular path with constant angular speed ω and has position vector $\vec{r}(t) = a\cos\omega t \cdot \hat{\imath} + a\sin\omega t \cdot \hat{\jmath}$.
Find the force acting on the particle and show that it is directed toward origin.



Angular speed: $\omega = \dfrac{d\theta}{dt}$

$x = a\cos\omega t$

$y = a\sin\omega t$



$\vec{r}(t) = a\cos\omega t \cdot \hat{\imath} + a\sin\omega t \cdot \hat{\jmath}$

$\vec{r}\,'(t) = -a\omega\sin\omega t \cdot \hat{\imath} + a\omega \cdot \cos\omega t \cdot \hat{\jmath}$

$$\vec{r}\,''(t) = -a\omega^2\cos\omega t \cdot \hat{\imath} - a\omega^2\sin\omega t \cdot \hat{\jmath} = -\omega^2\vec{r}(t)$$

$$\vec{F} = m\vec{a} = m\vec{r}\,''(t) = -\underbrace{m\omega^2}_{>0}\vec{r}(t)$$

$\Rightarrow$ Force $\vec{F}$ is directed opposite to $\vec{r}(t)$, i.e. toward the center.
We have centripetal / center-seeking force.

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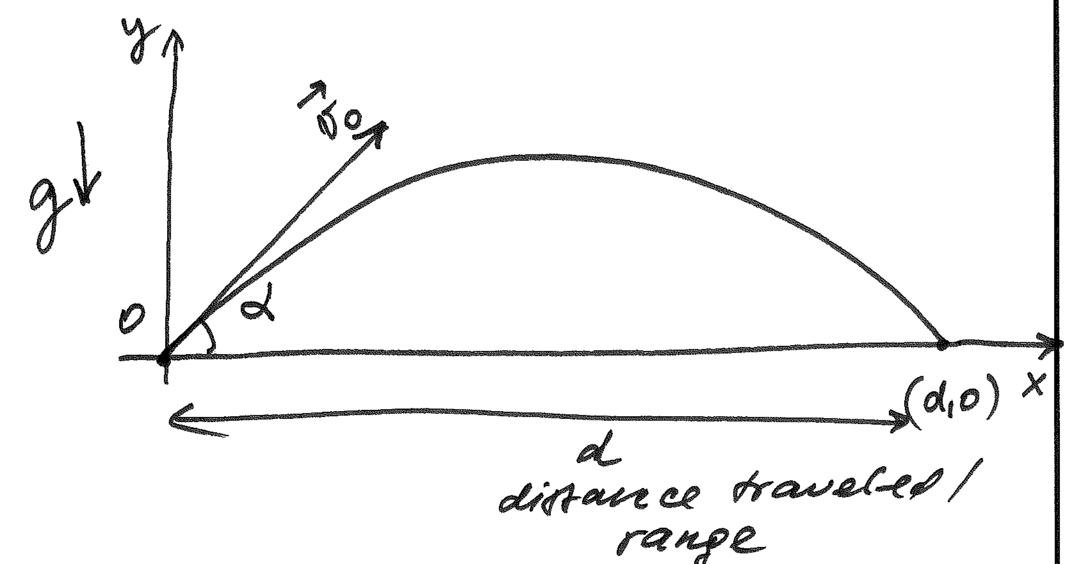
Ex A projectile is fired with angle of elevation α and initial velocity $\vec{v}_0$. Assuming that air resistance is negligible, and the only external force is due to gravity, find position function $\vec{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance traveled)?



$\vec{g} = -g\hat{j}$

g: acceleration due to gravity

$\vec{F} = m\vec{a}$ $\qquad$ $\vec{a} = \vec{g} = -g\hat{j}$

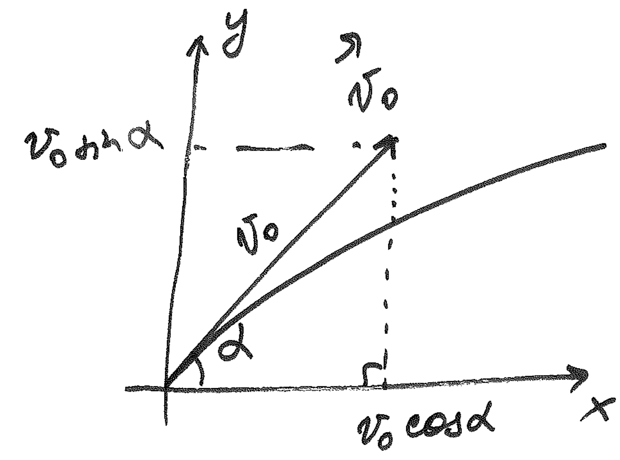
$$\vec{v}\,' = \vec{a} \Rightarrow \vec{v} = \int \vec{a}\, dt = \int -g\hat{j}\, dt = -gt\cdot\hat{j} + \vec{C}_1$$

at $t=0$, $\vec{v}\big|_{t=0} = \vec{v}_0 \rightarrow \underbrace{\vec{v}\big|_{t=0}}_{\vec{v}_0} = -g\cdot 0\cdot\hat{j} + \vec{C}_1 \Rightarrow \vec{C}_1 = \vec{v}_0$

$$\therefore \boxed{\vec{v} = -gt\cdot\hat{j} + \vec{v}_0}$$

$$\vec{r}\,'(t) = \vec{v}(t) \Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt = \int \left(-gt\cdot\vec{j} + \vec{v_0}\right) dt =$$

$$= -g\frac{t^2}{2}\vec{j} + \vec{v_0}\cdot t + \vec{C_2}$$



at $t=0$, $\vec{r}(0) = \vec{0} \Rightarrow \vec{C_2} = \vec{0}$

$$\therefore \boxed{\vec{r}(t) = -\frac{gt^2}{2}\vec{j} + \vec{v_0}\cdot t}$$

Let $v_0 = |\vec{v_0}|$: initial speed

$$\Rightarrow \vec{v_0} = v_0 \cos\alpha\cdot\vec{i} + v_0 \sin\alpha\cdot\vec{j}$$

$$\Rightarrow \vec{r}(t) = -\frac{gt^2}{2}\vec{j} + v_0 t \cos\alpha\cdot\vec{i} + v_0 t \sin\alpha\cdot\vec{j}$$

$$\boxed{\vec{r}(t) = v_0 t \cos\alpha\cdot\vec{i} + \left(v_0 t \sin\alpha - \frac{gt^2}{2}\right)\vec{j}}$$

$\vec{r}(t) = \langle x(t), y(t)\rangle$

$$\therefore \boxed{x(t) = v_0 t \cos\alpha, \quad y(t) = v_0 t \sin\alpha - \frac{gt^2}{2}}$$

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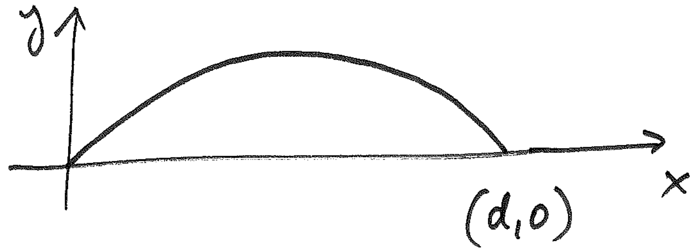

at the ground, $y=0$

$\Rightarrow v_0 t \sin\alpha - \frac{gt^2}{2} = 0$

$t\left(v_0 \sin\alpha - \frac{gt}{2}\right) = 0$

$t_1 = 0$ $\quad t_2 = \frac{2v_0}{g}\sin\alpha$

touchdown time

To find range d,

substitute t_2 in $x(t)$:

$x(t_2) = v_0 \cdot \frac{2v_0}{g}\sin\alpha \cdot \cos\alpha$

$2\sin\alpha \cdot \cos\alpha = \sin 2\alpha$

$x(t_2) = \boxed{\frac{v_0^2}{g}\sin 2\alpha = d}$: range

$d = d_{max}$ when $\sin 2\alpha = 1$ or $2\alpha = \frac{\pi}{2} \Rightarrow \boxed{\alpha = \frac{\pi}{4} = \alpha_{opt}}$

$\therefore \boxed{d_{max} = \frac{v_0^2}{g}}$

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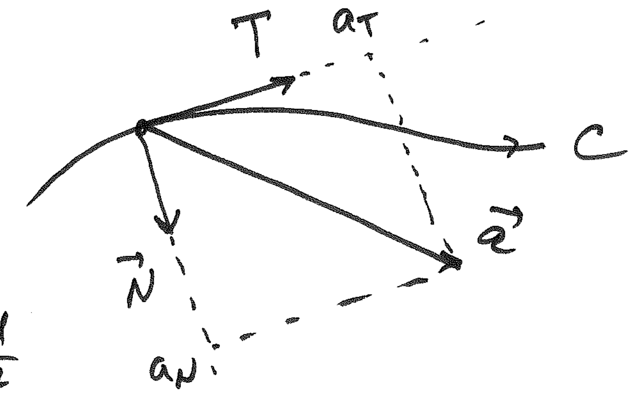
Tangential and Normal Components of Acceleration

Sometimes it is useful to write acceleration vector as

$$\vec{a} = a_T \cdot \vec{T} + a_N \cdot \vec{N}$$



$v = |\vec{v}|$: speed

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{v}}{v} \Rightarrow \boxed{\vec{v} = v\,\vec{T}} \qquad \frac{d}{dt}$$

$$\frac{d\vec{v}}{dt} = \frac{dv}{dt} \cdot \vec{T} + v \cdot \frac{d\vec{T}}{dt} = v' \cdot \vec{T} + v \cdot \vec{T}\,' \qquad (1)$$

$\frac{d\vec{v}}{dt} = \vec{a}$

$\vec{T} \perp \vec{T}\,'$ and $\vec{T}\,' \parallel \vec{N}$ and $\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} \Rightarrow \boxed{\vec{T}\,' = |\vec{T}\,'| \cdot \vec{N}}$ (2)

$$\underset{\text{curvature}}{K} = \frac{|\vec{T}\,'|}{\underbrace{|\vec{r}\,'|}_{v}} \Rightarrow \boxed{|\vec{T}\,'| = K v}$$

substitute in (2)

$$\Rightarrow \boxed{\vec{T}\,' = K v \cdot \vec{N}}$$

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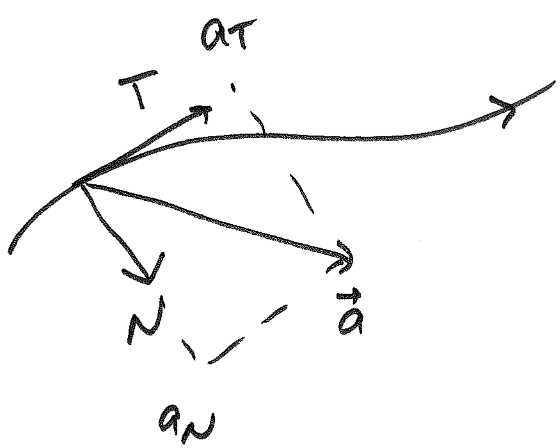
Substitute $\vec{T}' = \kappa v \cdot \vec{N}$ in (1):

$$\vec{a} = \underbrace{v'}_{a_T} \cdot T + \underbrace{\kappa v^2}_{a_N} \cdot \vec{N} \qquad \text{or} \qquad \vec{a} = a_T \vec{T} + a_N \cdot \vec{N}$$

where $a_T = v'$

$a_N = \kappa v^2$



$\vec{T}$ points in the direction of motion

$\vec{N}$ points in the direction curve is turning

We can find expressions for a_T and a_N using $\vec{r}, \vec{r}'$ and $\vec{r}''$:

$$\boxed{a_T = v' = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}} \quad \text{and} \quad \boxed{a_N = \kappa v^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}}$$

Indeed,

$$\vec{v} = v \cdot \vec{T}$$

$$\vec{v} \cdot \vec{a} = v \cdot \vec{T} \cdot \left(v' \vec{T} + \kappa v^2 \vec{N} \right) = v v' \cdot \underbrace{(\vec{T} \cdot \vec{T})}_{\substack{= \\ |\vec{T}|^2 = 1}} + \kappa v^3 \underbrace{(\vec{T} \cdot \vec{N})}_{\substack{= \\ 0 \\ \text{since } \vec{T} \perp \vec{N}}} = v v'$$

$$\Rightarrow \underset{\substack{= \\ a_T}}{v'} = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

$$a_N = \kappa v^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \cdot |\vec{r}'(t)|^2 = \frac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t)|}$$