Lecture 16

14.2 Limits and Continuity

Consider

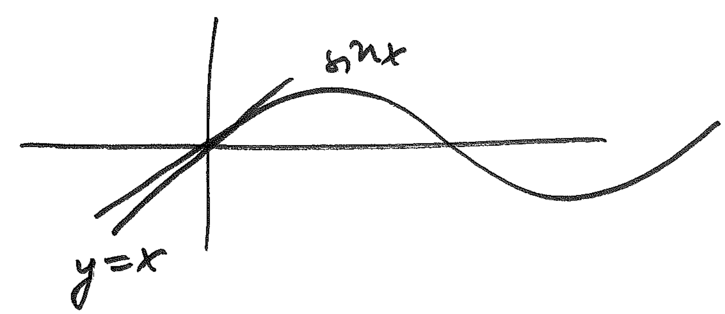
$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \qquad\qquad g(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

Q How do f and g behave as $(x,y) \to (0,0)$?

Tables 1 and 2 on pg. 944 of numerical values (see) of f around $(0,0)$ suggest that $f \to 1$ as $(x,y) \to (0,0)$, whereas there is no limit for g as $(x,y) \to (0,0)$.

<u>Note</u> $\sin x \sim x$ for small x

$$\Rightarrow \frac{\sin x}{x} \sim \frac{x}{x} = 1$$



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Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

Table 1 Values of $f(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that as (x, y) approaches $(0, 0)$, the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \qquad \text{and} \qquad \lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

In general, we use the notation

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f. In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y)\to(a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L \qquad \text{and} \qquad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Notice that $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L, and $\sqrt{(x - a)^2 + (y - b)^2}$ is the distance between the point (x, y) and the point (a, b). Thus Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by

We write

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$$

We use notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

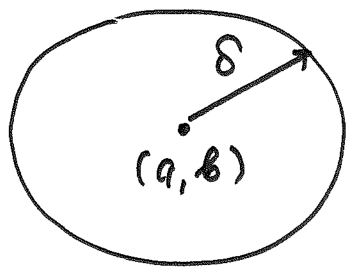
to indicate whether values of f become very close to L as (x,y) approaches (a,b).

(x,y) remains in domain D of f.

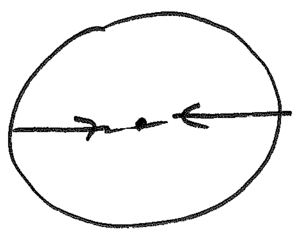
Def Function $f(x,y)$ has limit L as $(x,y) \in D$ approach (a,b) along any path and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.



(x,y) can approach (a,b) along ANY PATH!



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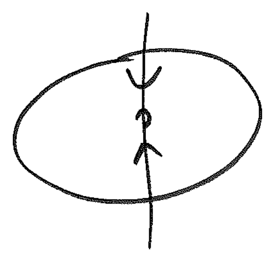

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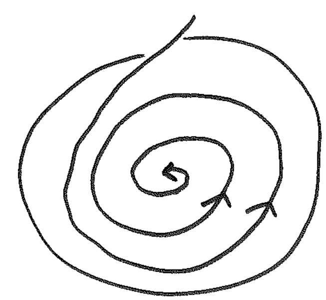

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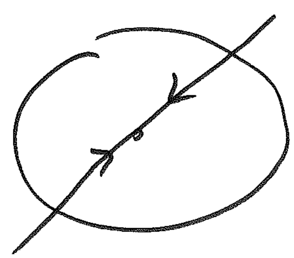

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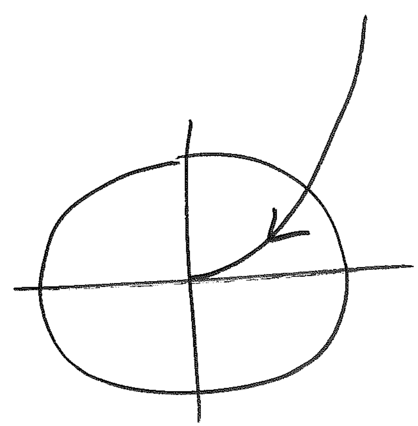

Note

If $\underbrace{\lim_{\substack{(x,y)\to(a,b)\\ \text{along path 1}}} f(x,y)}_{=L_1} \neq \underbrace{\lim_{\substack{(x,y)\to(a,b)\\ \text{along path 2}}} f(x,y)}_{=L_2}$

$\exists$: exist

$\nexists$: does not exist

$\forall$: any

then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist or write $\nexists$

if $L_1 \neq L_2 \Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) \nexists$

Ex Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

We need to show

$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 : \quad 0 < \sqrt{(x-\underset{a}{0})^2 + (y-\underset{b}{0})^2} < \delta \Rightarrow \Big| \underbrace{\frac{xy}{\sqrt{x^2+y^2}}}_{f(x,y)} - \underset{L}{0} \Big| < \varepsilon$

$0 < \boxed{\sqrt{x^2+y^2} < \delta}$

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$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \frac{|xy|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2y^2}}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2x^2y^2}}{\sqrt{x^2+y^2}} \leq$$

$$(x^2+y^2)^2 = x^4 + 2x^2y^2 + y^4$$

$$\leq \frac{1}{\sqrt{2}}\frac{\sqrt{x^4+2x^2y^2+y^4}}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{(x^2+y^2)^2}}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}}\frac{x^2+y^2}{\sqrt{x^2+y^2}} =$$

need

$$= \frac{1}{\sqrt{2}}\underbrace{\sqrt{x^2+y^2}}_{<\delta} < \varepsilon \qquad \frac{1}{\sqrt{2}}\delta = \varepsilon \Rightarrow \delta = \sqrt{2}\,\varepsilon$$

This shows that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$