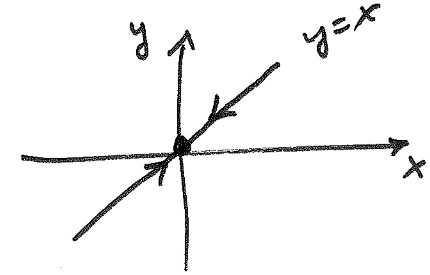
Lecture 17

Limits (Cont'd)

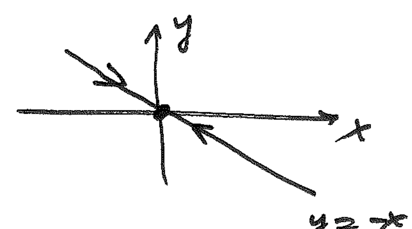
Ex Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy\cos y}{x^2+y^2}$ does not exist.

Path 1 : $y=x$ or $x=y$



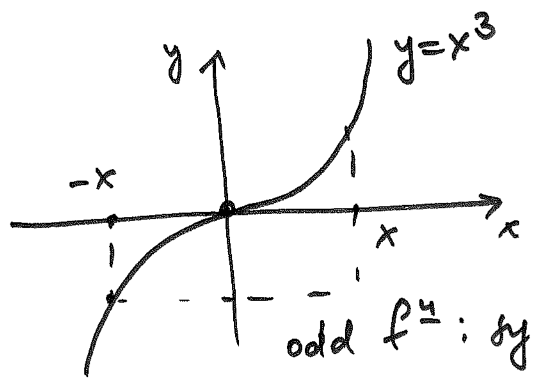
$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2} = \lim_{x\to 0} \frac{x^2\cos x}{\underbrace{x^2+x^2}_{=2x^2}} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2} = L_1$$

Path 2 : $x=-y$ or $y=-x$

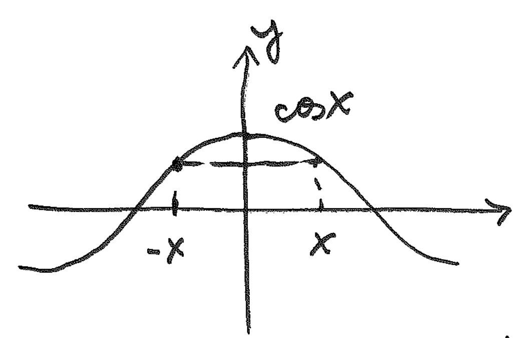


$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2} = \lim_{x\to 0} \frac{-x^2\cos(-x)}{x^2+(-x)^2} = \lim_{x\to 0} \frac{-x^2\cdot\cos x}{2x^2} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2} = L_2$$

$\cos(-x)=\cos x$: cos is even f^{n}



odd f^{n} : symmetry about origin



even: symmetric about y-axis

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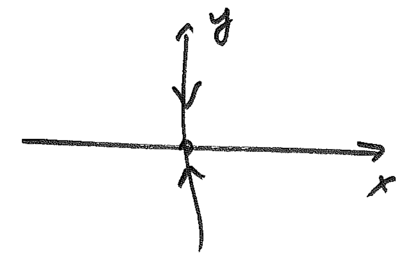
$L_1 \neq L_2 \Rightarrow \lim\limits_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2}$ does not exist or $\nexists$

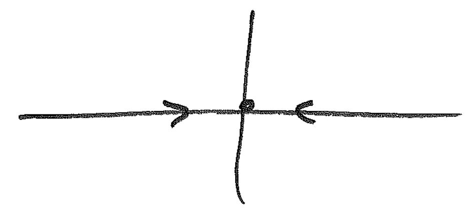
<u>Note</u> if $L_1 = L_2 \Rightarrow$ does <u>NOT</u> imply that limit exists

<u>Path 3</u>: $x=0$: y-axis



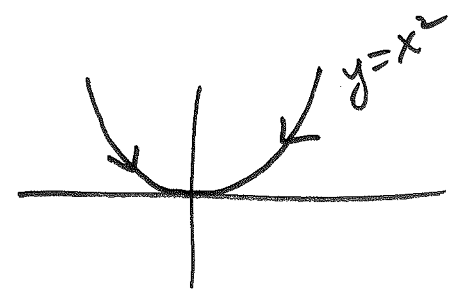
$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2} = \lim_{\substack{y\to 0\\ y\neq 0}} \frac{0\cdot y\cos y}{0+y^2} = \lim_{y\to 0} \frac{0}{\underbrace{y^2}_{\neq 0}} = 0 = L_3$$

<u>Path 4</u>: $y=0$: x-axis



$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot 0\cos 0}{x^2+0} = \lim_{x\to 0} \frac{0}{\underbrace{x^2}_{\neq 0}} = 0 = L_4$$

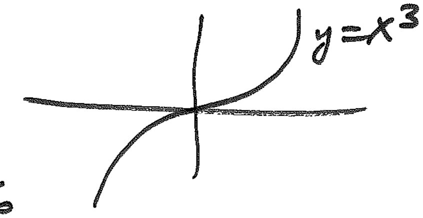
$\Rightarrow L_3 = L_4$ but limit $\nexists$



<u>Path 5</u>: $y=x^2$: parabola

$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot x^2\cos x^2}{x^2+x^4} = \lim_{x\to 0} \frac{x\cos x^2}{1+x^2} = \frac{0}{1} = 0 = L_5$$

Path 6 : $y = x^3$: cubic parabola



$$\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot x^3 \cos x^3}{x^2+x^6} = \lim_{x\to 0}\frac{x^2\cos x^3}{1+x^4} = \frac{0}{1} = L_6$$

Def Function $f(x,y)$ is continuous (CTS) at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Def $f(x,y)$ is continuous in domain D if it is continuous at every pt $(x,y) \in D$.

Note f is CTS : small changes in (x,y) result in small changes in f

Sum, difference, product and ratio (denominator $\neq 0$) of CTS functions are also continuous.

Def a polynomial of two variables is a sum of terms

$cx^n y^m$, c : const, $n, m = 0, 1, 2, \dots$

Def Rational function is a ratio of two polynomials.

Ex $f(x,y) = x^2y^3 - xy + 5x - 7$: polynomial

Ex $g(x,y) = \dfrac{x + x^3 y}{y^2 + x^7}$: rational function

Polynomials are cts f^{ns}.

Rational functions are cts where denom. $\neq 0$.

Ex Evaluate

$$\lim_{(x,y)\to(1,2)} (\underbrace{5xy^2 - 3 + x^3y^4 - x}_{\substack{\text{polynomial} \Rightarrow \text{it is cts everywhere} \\ \text{including at } (x,y)=(1,2)}}) = 5\cdot1\cdot2^2 - 3 + 1^3\cdot 2^4 - 1 = 32$$

$\Rightarrow \lim_{(x,y)\to(1,2)} = \text{value}\big|_{(1,2)}$

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Ex $f(x,y) = \sin(xy^2)$: CTS f^n as a composition of $\sin(\ldots)$ and polynomial, both CTS f^{ns}, i.e. composition of two CTS functions.

Ex Where is the function $f(x,y) = \dfrac{xy\cos y}{x^2+y^2}$ continuous?

$f(x,y)$ is discontinuous (has no limit) at $(0,0)$ since it is not defined there. Moreover, we just showed that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore$ $f(x,y)$ is continuous everywhere except at $(0,0)$

Ex

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is function continuous?

For $(x,y) \neq (0,0)$, $f(x,y)$ is CTS as a ratio of CTS functions.

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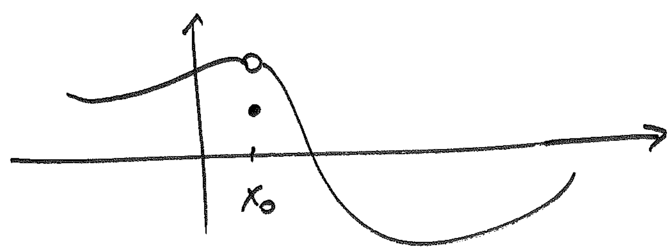
Last time we showed using ε-δ def that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$$

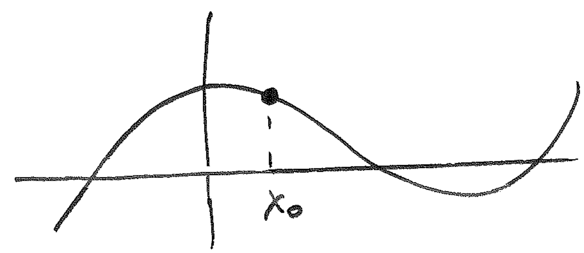
$\therefore$ $f(x,y)$ is continuous everywhere

<u>Note</u> Consider 1D case

Ex



NOT cts at x_0



cts at x_0

Ex $\frac{\sin x}{x}$ not defined at $x=0$

but $\lim_{x\to 0} \frac{\sin x}{x} \underset{\text{rule}}{\overset{\text{l'Hop.}}{=}} \lim_{x\to 0} \frac{\cos x}{1} = 1$

"$\frac{0}{0}$"

$\Rightarrow$ $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is cts everywhere

Note We can define limit for functions of 3 variables in a similar way.

Def $f(x,y,z)$ is continuous at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$$

14.3 Partial Derivatives

Consider $f(x,y)$: function of two variables x and y

Let $y = b$: fixed $\Rightarrow$ $f(x,b) \equiv g(x)$: f^n of x

If $g(x)$ is differentiable at $x = a \Rightarrow$ we can evaluate $g'(a) = \frac{dg}{dx}(a)$. This derivative is called a partial derivative of f wrt x.

$$g'(a) = f_x(a,b) = \frac{\partial f}{\partial x}(a,b)$$

Recall

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$\Rightarrow \quad f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly, to get a partial derivative of $f(x,y)$ wrt y, we fix $x=a$ $(\Rightarrow G(y) = f(a,y))$. If $G(y)$ is differentiable at $y=b$ $\Rightarrow$

$$G'(b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} \equiv \frac{\partial f}{\partial y}(a,b)$$

In general,

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$