Lecture 18

Ex Find f_x and f_y of $f(x,y) = x^4 + 3xy^3 - 2x^2y^2$

$f_x = 4x^3 + 3y^3 - 4xy^2$; $\quad f_y = 0 + 3x \cdot 3y^2 - 2x^2 \cdot 2y$

$= 9xy^2 - 4x^2y$

Ex Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$yz = \ln(x+z)$$

Here: x, y: independent variables

$z = z(x,y)$: dependent variables

$\frac{\partial x}{\partial x} = 1 \qquad \frac{\partial y}{\partial x} = 0$

Take partial derivative wrt x of both sides of eq^n

$\frac{\partial}{\partial x}$: $\quad yz = \ln(x+z)$

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$$\frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}\left(\ln(x+z)\right)$$

$$\frac{\partial}{\partial x}(yz) = \cancel{\frac{\partial y}{\partial x}}^{\,0} \cdot z + y\,\frac{\partial z}{\partial x}$$

product rule

$$\frac{\partial}{\partial x}\left(\ln(x+z)\right) = \frac{1}{x+z}\cdot\frac{\partial}{\partial x}(x+z) = \frac{1}{x+z}\left(\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial z}{\partial x}\right)$$

$$\therefore \quad y\,\frac{\partial z}{\partial x} = \frac{1}{x+z}\left(1+\frac{\partial z}{\partial x}\right)$$

Solve for $\frac{\partial z}{\partial x}$:

$$y\,\frac{\partial z}{\partial x} = \frac{1}{x+z} + \frac{1}{x+z}\cdot\frac{\partial z}{\partial x}$$

$$\left(y - \frac{1}{x+z}\right)\frac{\partial z}{\partial x} = \frac{1}{x+z} \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{\frac{1}{x+z}}{y-\frac{1}{x+z}} = \frac{1}{y(x+z)-1}$$

$$\therefore \quad \boxed{\frac{\partial z}{\partial x} = \frac{1}{y(x+z)-1}}$$

$\frac{\partial}{\partial y}$: $\quad yz = \ln(x+z)$

$$\underbrace{\frac{\partial y}{\partial y}}_{=1} \cdot z + y\frac{\partial z}{\partial y} = \frac{1}{x+z} \cdot \frac{\partial}{\partial y}(x+z)$$

$$z + y\frac{\partial z}{\partial y} = \frac{1}{x+z}\left(\cancelto{0}{\frac{\partial x}{\partial y}} + \frac{\partial z}{\partial y}\right)$$

x, y are indep. variables

$$z + y\frac{\partial z}{\partial y} = \frac{1}{x+z} \cdot \frac{\partial z}{\partial y}$$

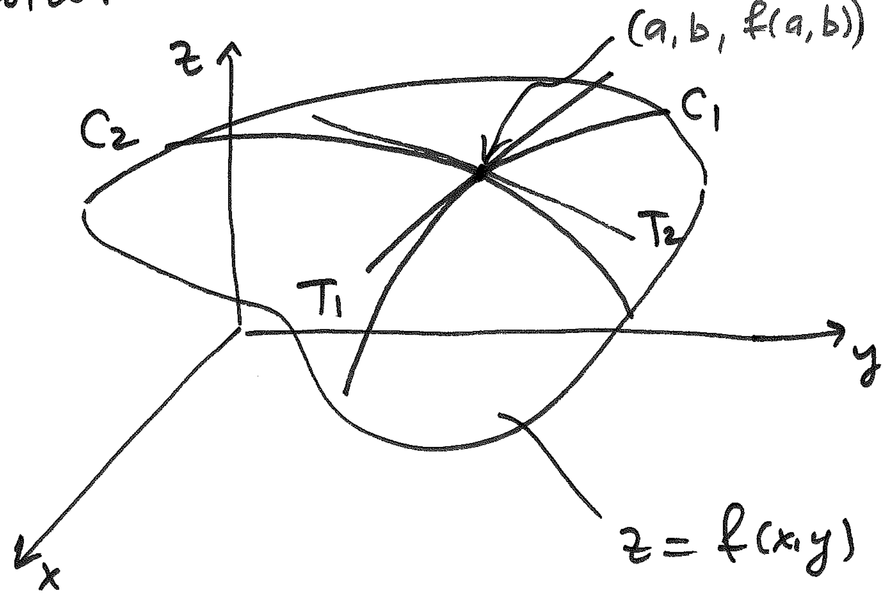
Solve for $\frac{\partial z}{\partial y}$:

$$\left(y - \frac{1}{x+z}\right)\frac{\partial z}{\partial y} = -z \Rightarrow \frac{\partial z}{\partial y} = \frac{z}{\frac{1}{x+z} - y} = \frac{z(x+z)}{1-y(x+z)}$$

$$\therefore \boxed{\frac{\partial z}{\partial y} = \frac{z(x+z)}{1-y(x+z)}}$$

Geometric interpretation of partial derivatives

Consider a function $z = f(x,y)$. Its graph is a surface.



Fix $y = b \Rightarrow$ we take a cross-section of the surface w/ plane $y = b \Rightarrow$ get curve C_1 $\quad g(x) = f(x, b)$

Slope of tangent line T_1 of curve C_1 at $(a, b, f(a,b))$ is

$$g'(a) = f_x(a, b)$$

Similarly, if we fix $x = a \Rightarrow$ cross-section of surface w/ plane $x = a$ is curve C_2. The slope of tangent line to C_2 at $(a, b, f(a,b))$ is $f_y(a, b)$.

Hence, partial derivatives $f_x(a,b)$ and $f_y(a,b)$ are slopes of tangent lines at pt $(a, b, f(a,b))$ of traces C_1 and C_2 w/ plane $y=b$ and $x=a$.

<u>Another interpretation:</u> $\frac{\partial f}{\partial x}$ is the rate of change of $f(x,y)$ wrt x. Similarly for $\frac{\partial f}{\partial y}$.

<u>Higher order derivatives</u>

Consider function $f(x,y)$. $f_x(x,y)$, $f_y(x,y)$ are also functions of x and y, in general. If f_x, f_y are differentiable, we can consider derivatives of f_x, f_y wrt x and y.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$: 2^{nd} order partial derivative wrt x

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$: mixed 2^{nd} order derivative

<u>Note</u>: $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$: another mixed 2nd order derivative

$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$: 2nd order derivative wrt y

Ex $f(x,y) = \ln(3x+5y)$

$f_x = \frac{3}{3x+5y}$ $\qquad f_y = \frac{5}{3x+5y}$

$\left(\frac{1}{\dots}\right)_x$ $\qquad \frac{\partial}{\partial x}(3x+5y)$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{3}{3x+5y}\right) = 3 \cdot \left(-\frac{1}{(3x+5y)^2}\right) \cdot 3 =$

$= -\frac{9}{(3x+5y)^2}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{3}{3x+5y}\right) = -\frac{3}{(3x+5y)^2} \cdot 5 = -\frac{15}{(3x+5y)^2}$

$\frac{\partial}{\partial y}(3x+5y)$

$(x^n)' = nx^{n-1}$ $\quad n=-1$ $\qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{5}{3x+5y}\right) = -\frac{5}{(3x+5y)^2} \cdot \underset{\frac{\partial}{\partial x}(3x+5y)}{3} =$$

$$= -\frac{15}{(3x+5y)^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{5}{3x+5y}\right) = -\frac{5}{(3x+5y)^2} \cdot \underset{\frac{\partial}{\partial y}(3x+5y)}{5} =$$

$$= -\frac{25}{(3x+5y)^2}$$

Note: $f_{xy} = f_{yx}$

Clairaut's Thm Suppose $f(x,y)$ is defined on some disk D that contains pt (a,b) and partial derivatives f_x and f_y are both continuous on D. Then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

Partial Differential Equations

(Math 480 PDEs)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad : \quad \text{Laplace's equation in 2D}$$

$u = u(x,y)$

a function that satisfies Laplace's eqn is called a harmonic function.

Applications: heat conduction (steady state), fluid flow, problem w/ applied electric field

Ex Show that $u(x,y) = x^2 - y^2$ is a solution of Laplace's eqn

$$u_{xx} + u_{yy} = 0$$

$u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$, $u_{yy} = -2$

$\Rightarrow u_{xx} + u_{yy} = 2 + (-2) \equiv 0$ for all x ✓

$\therefore$ $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$

3D: $u_{xx} + u_{yy} = u_{zz} = 0$: Laplace's eq$^{=}$ in 3D

<u>Notation:</u> $\Delta u = 0$ or $\nabla^2 u = 0$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$