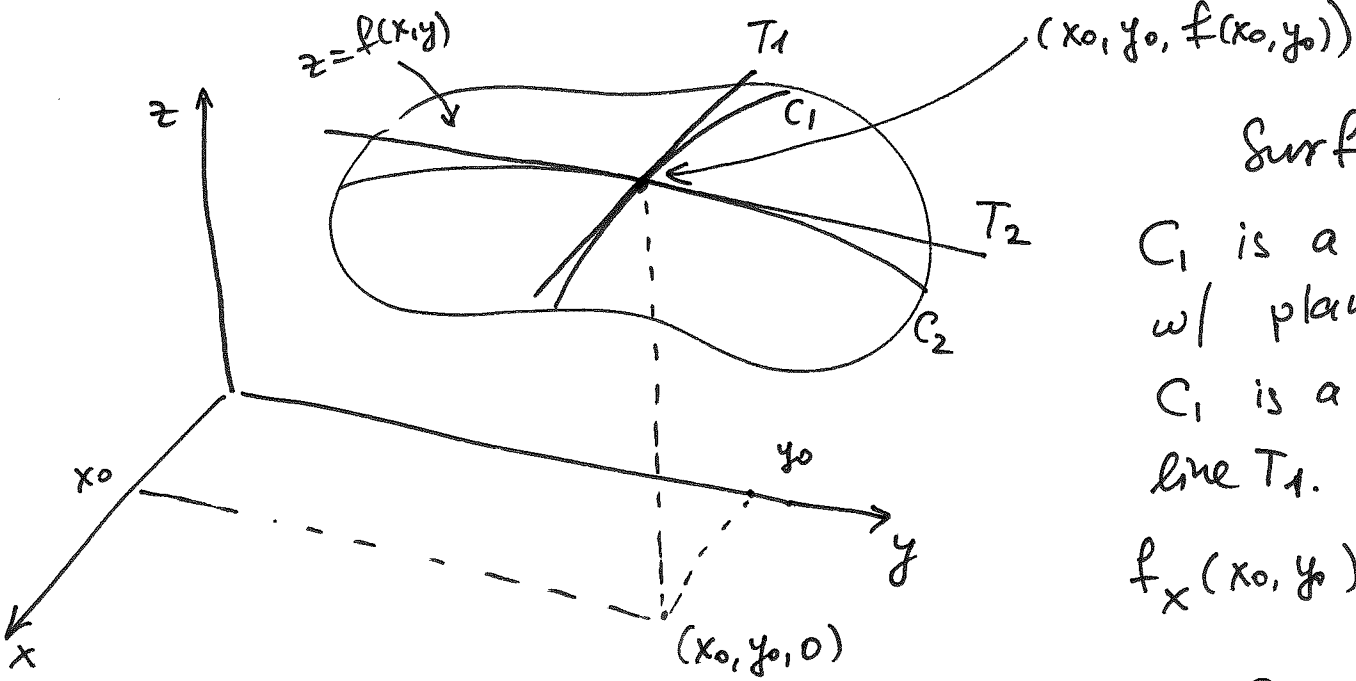
Lecture 19

14.4 Tangent Planes



Surface $z=f(x,y)$.

C_1 is a trace of surface w/ plane $y=y_0$

C_1 is a curve w/ tangent line T_1. T_1 has the slope $f_x(x_0,y_0)$ at $(x_0, y_0, f(x_0,y_0))$.

C_2 is a trace of surface w/ plane $x=x_0$. C_2 has tangent line T_2 whose slope is $f_y(x_0,y_0)$ at $(x_0, y_0, f(x_0,y_0))$.

Plane that contains tangent lines T_1 and T_2 is called <u>tangent plane</u>.

Note Any curve on the surface $z = f(x,y)$ through pt $(x_0, y_0, f(x_0,y_0))$ has tangent line in tangent plane.

Q What is the equation of this tangent plane?

$\frac{1}{C}\Big|$ $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$: eqⁿ of the plane through pt (x_0, y_0, z_0) and normal vector $\vec{n} = \langle A, B, C\rangle$

$$\Rightarrow z - z_0 = \underbrace{-\frac{A}{C}}_{= a}(x-x_0) \underbrace{-\frac{B}{C}}_{= b}(y-y_0)$$

i.e.

$$z - z_0 = a(x-x_0) + b(y-y_0) \qquad (1)$$

Tangent line T_1 belongs to this tangent plane. We fix $y = y_0$.
Substitute in (1):

$$z - z_0 = a(x-x_0) \quad : \quad \text{tangent line } T_1$$

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$\Rightarrow$ slope of T_1 is a but it is also $f_x(x_0, y_0)$

$$\Rightarrow \boxed{a = f_x(x_0, y_0)}$$

Tangent line T_2: $x = x_0$. Substitute in (1):

$z - z_0 = b(y - y_0)$: tangent line T_2

Slope of T_2 is b but also $f_y(x_0, y_0)$.

$$\Rightarrow \boxed{b = f_y(x_0, y_0)}$$

$z_0 = f(x_0, y_0)$

Hence,

(*) $$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

eq^n of tangent plane to $z = f(x, y)$ at $(x_0, y_0, \underbrace{f(x_0, y_0)}_{= z_0})$

We can write this eq^n as

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Normal vector is $\vec{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$

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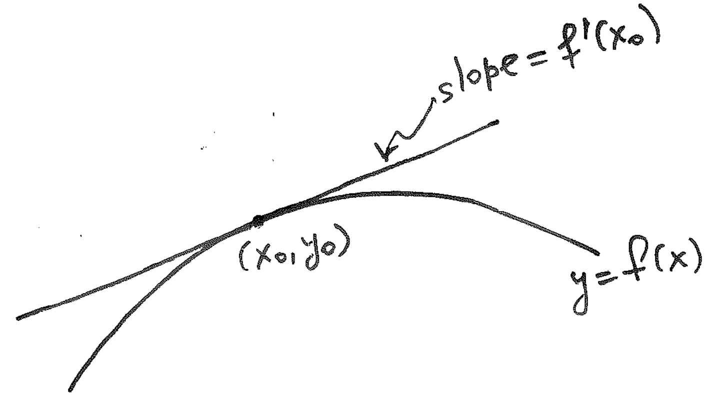
Recall 1D case: $y = f(x)$

Tangent line:

$$\boxed{y - y_0 = f'(x_0)(x - x_0)}$$



Compare this eqn w/ (*).

Ex Find tangent plane to surface

$z = 9x^2 + y^2 + 6x - 3y + 5$

at $(1, 2, 18)$.
$\ x_0 = 1,\ y_0 = 2,\ z_0 = 18$

$z(1,2) = 9 \cdot 1^2 + 2^2 + 6 \cdot 1 - 3 \cdot 2 + 5 = 18$ ✓

$f(x,y) = 9x^2 + y^2 + 6x - 3y + 5$

$f_x = 18x + 6$

$f_y = 2y - 3$

$f_x(1,2) = f_x\Big|_{\substack{x=1\\y=2}} = 18 \cdot 1 + 6 = 24$

$f_y(1,2) = (2y - 3)\Big|_{\substack{x=1\\y=2}} = 2 \cdot 2 - 3 = 1$

Tangent plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 18 = 24(x - 1) + 1 \cdot (y - 2)$$

$\boxed{z = 24x + y - 8}$ tangent plane at $(1, 2, 18)$

Linear approximation

From previous example,

$L(x,y) = 24x + y - 8$: good approximation of surface for points close to $(1, 2, 18)$

we write

$$f(x,y) \approx 24x + y - 8$$

$L(x,y)$: linear approximation of f or linearization

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$x_0 = 1,\ y_0 = 2$

Ex $\underbrace{f(1.1, 1.95)}_{\text{close to } (1,2)} \overset{?}{\approx} 24\cdot(1.1) + 1.95 - 8 = \boxed{20.35}$

$f(x,y) = 9x^2 + y^2 + 6x - 3y + 5$

$\underbrace{f(1.1, 1.95)}_{\text{exact value}} = 9\cdot(1.1)^2 + (1.95)^2 + 6\cdot(1.1) - 3\cdot(1.95) + 5 = \boxed{20.4425}$

Note: value 20.35 is close to $f_{exact}(1.1, 1.95) = 20.4425$

Now, take pt away from $(1,2)$:

$f(5,-1) \overset{?}{\approx} 24\cdot 5 + (-1) - 8 = 111$

but exact value $f_{exact}(5,-1) = 264$

In general,

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

is equation of the tangent plane at $(a, b, f(a,b))$.

Then

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

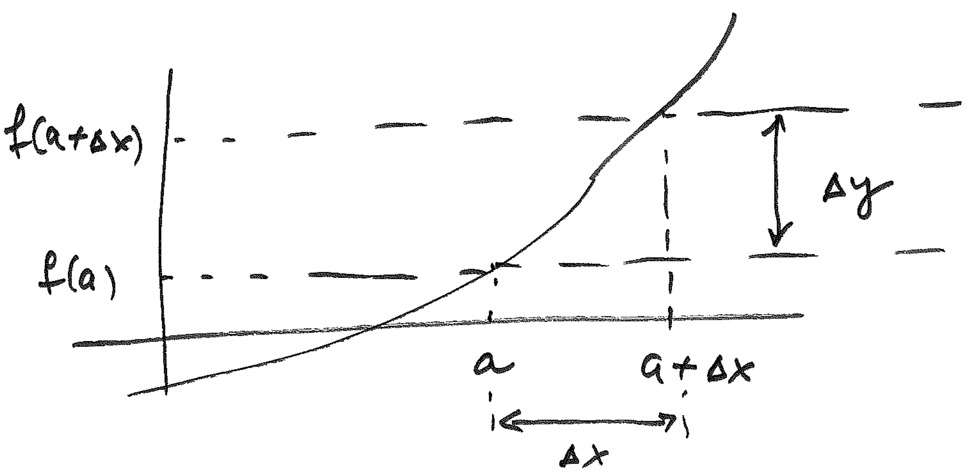
is <u>linear approximation</u> / <u>linearization</u> of $f(x,y)$ near (a,b).

i.e.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Differentiable functions / Differentials

1D case : $y = f(x)$



$x = a$ $\quad\quad$ $x = a + \Delta x$

Δx : increment

$$\Delta y = f(a+\Delta x) - f(a) =$$

$$= y(a+\Delta x) - y(a)$$

Def Function $y=f(x)$ is differentiable at $x=a$ if one can write

$$\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

Note $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$

Consider $z = f(x,y)$

$x: \; a \longrightarrow a+\Delta x$

$y: \; b \longrightarrow b+\Delta y$

$$\Delta z = f(a+\Delta x,\, b+\Delta y) - f(a,b)$$

Def Function $z=f(x,y)$ is differentiable at (a,b) if

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$.

These defs are not very convenient to use. Instead, we will use the following result.

Thm

If f_x, f_y exist near (a, b) and continuous at (a, b), then $f(x,y)$ is differentiable at (a, b).