Lecture 20

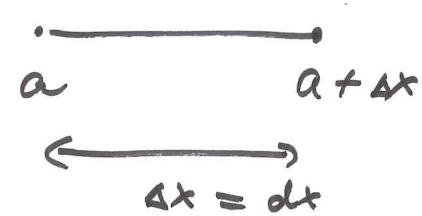
Differentials (S 14.4 cont'd)

1D $y = f(x)$

We define

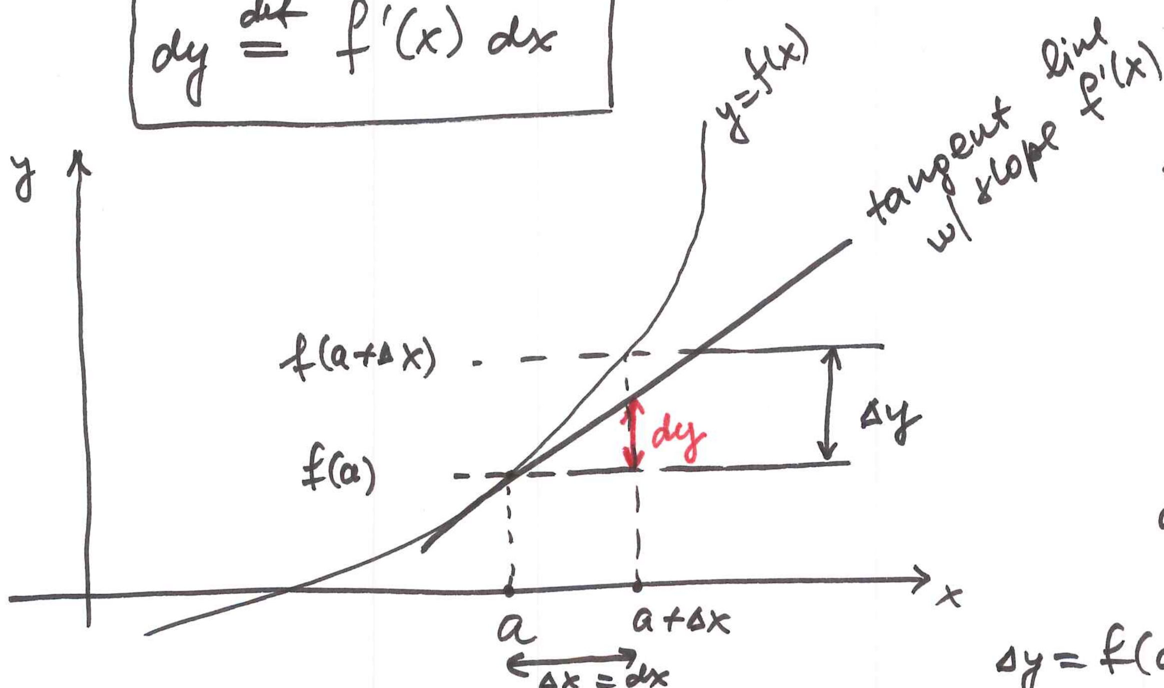
dx: differential of independent variable

$dx = \Delta x$



and let

$$\boxed{dy \overset{\text{def}}{=} f'(x)\, dx}$$



Δy: true displacement of $f(x)$ as x changes from a to $a + \Delta x$, i.e. true displacement of y

dy: displacement of y along tangent line

$\Delta y = f(a + \Delta x) - f(a)$

2D Consider $z = f(x,y)$

$a \to a + \Delta x \equiv x$
$b \to b + \Delta y \equiv y$

We set $dx = \Delta x$, $dy = \Delta y$

independent differentials

We define

$$\boxed{dz = f_x(a,b)\,dx + f_y(a,b)\,dy}$$: differential of dependent variable z

Now, if we set $dx = \Delta x = x - a$, $dy = \Delta y = y - b$

$$\Rightarrow \boxed{dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

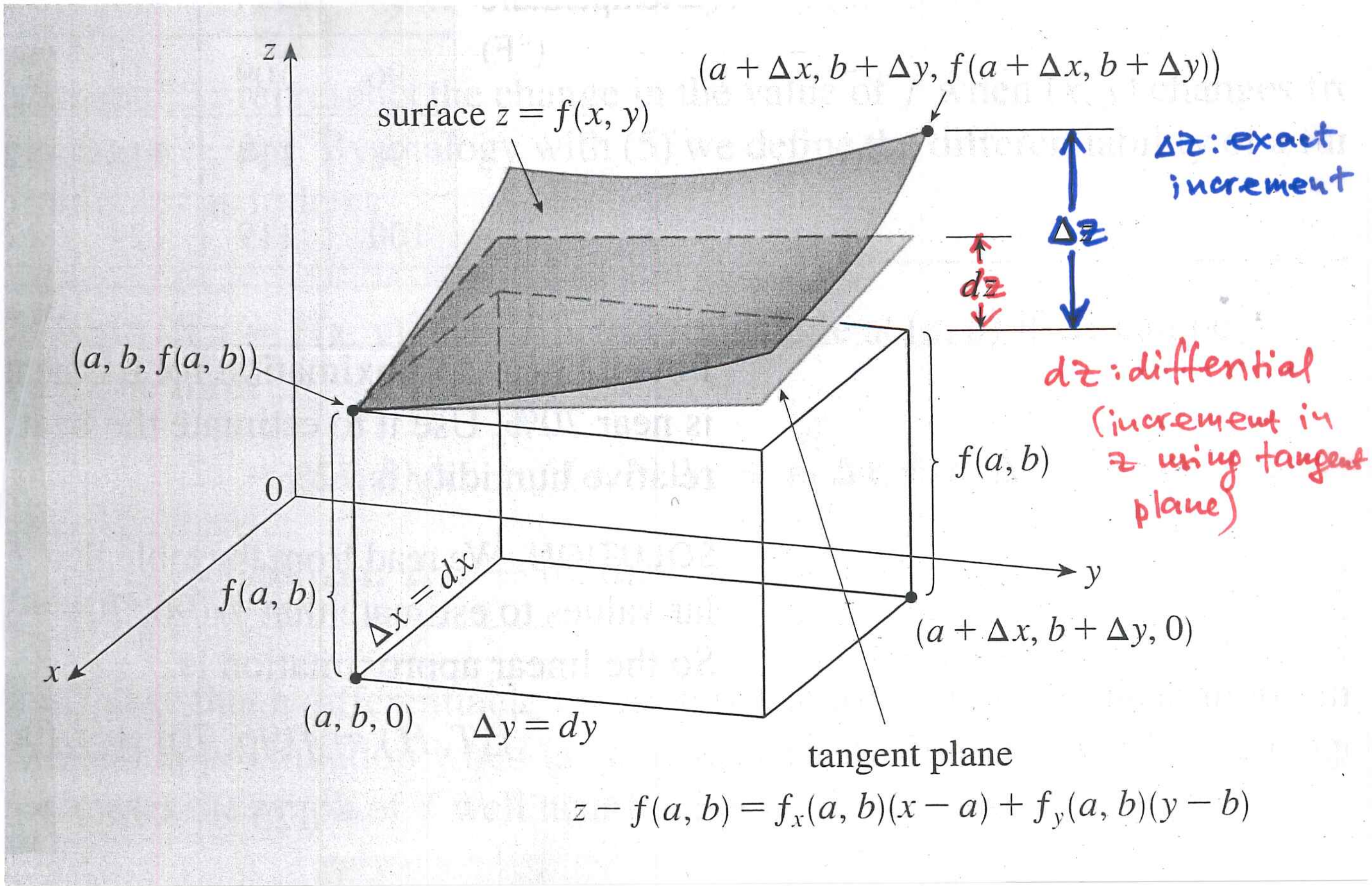
From last lecture:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

linearization of f near (a,b)

$$\Rightarrow \boxed{L(x,y) = f(a,b) + dz}$$

$\therefore \quad f(x,y) \approx L(x,y) \qquad \text{or} \qquad f(x,y) \approx f(a,b) + dz$



Y

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Ex Find differential of function $v = y \cos xy$

(a)

$$dv = v_x \, dx + v_y \, dy$$

$$v_x = y \cdot (-\sin(xy)) \cdot y = -y^2 \sin(xy)$$

$$v_y = 1 \cdot \cos(xy) + y \cdot (-\sin(xy)) \cdot x = \cos(xy) - xy \cdot \sin(xy)$$

$$\therefore \boxed{dv = -y^2 \sin(xy) \cdot dx + [\cos(xy) - xy \cdot \sin(xy)] \cdot dy}$$

Note: dv : function of x and y.

(b) If x changes from $x=1$ to $x=0.95$ and y changes from $y=2$ to $y=2.05$, compare Δv and dv.

$$\left\{ \begin{array}{l} a \to a + \Delta x \\ b \to b + \Delta y \end{array} \right.$$

Here $a = 1$ $\quad a + \Delta x = 0.95 \Rightarrow \Delta x = 0.95 - 1 = -0.05$

$b = 2$ $\quad b + \Delta y = 2.05 \Rightarrow \Delta y = 2.05 - 2 = 0.05$

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$$\Delta v = v(a+\Delta x, b+\Delta y) - v(a,b) = \underbrace{v(0.95, 2.05)}_{\text{substitute in } v} - v(1,2) =$$

$$= (2.05)\cdot\cos((0.95)\cdot(2.05)) - 2\cdot\cos(1.2) = -0.7541 - (-0.8323) =$$

$$= 0.0782 \Rightarrow \boxed{\Delta v = 0.0782}$$

$$dv\Big|_{(1,2)} = v_x(1,2)\cdot \underbrace{dx}_{=\Delta x} + v_y(1,2)\cdot \underbrace{dy}_{=\Delta y} = \left(-y^2\sin(xy)\right)\Big|_{\substack{x=1\\y=2}} \cdot \underbrace{(-0.05)}_{\Delta x} +$$

$$+\left[\cos(xy) - xy\cdot\sin(xy)\right]\Big|_{\substack{x=1\\y=2}} \cdot \underbrace{(0.05)}_{=\Delta y} = 0.1156 \Rightarrow \boxed{dv = 0.1156}$$

$$|\text{error}| = |\underbrace{\Delta v}_{\text{exact}} - \underbrace{dv}_{\text{approximation}}| = 0.0374 : \text{ absolute error}$$

$$\text{Percentage error} = \frac{\overbrace{|\Delta v - dv|}^{\text{abs. error}}}{\underbrace{|\Delta v|}_{\text{exact value}}} \cdot 100\% = \frac{0.0374}{0.0782}\cdot 100\% = \overbrace{0.478}^{\text{relative error}} \times 100\% =$$

relative error

$$= \boxed{47.8\%}$$

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Functions of Three and More Variables

Consider $w = f(x, y, z)$

x, y, z : independent variables

w: dependent variable

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

$\parallel$

$L(x,y,z)$: linear approximation of f near (a,b,c)

We define differential of w:

$f_x = w_x$

$w_y = f_y, \quad w_z = f_z$

$$dw = w_x\,dx + w_y \cdot dy + w_z \cdot dz$$

dx, dy, dz: differentials of independent variables

dw: differential of dependent variable

Note: $L(x, y, z) = f(a, b, c) + dw$

Increment in w (true/exact change of w):

$x \to x + \Delta x, \quad y \to y + \Delta y, \quad z \to z + \Delta z$

Then

$\Delta w = w(x + \Delta x, y + \Delta y, z + \Delta z) - w(x, y, z)$

Functions of more than 3 variables can be considered in a similar way.

14.5 Chain Rule

1D If $y = f(x)$ and $x = g(t)$, i.e. $y = f(g(t))$ is a function of t implicitly, then if f and g are differentiable, then

$$\frac{dy}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}$$

2D

I Now, let $z = f(x,y)$ and $x = g(t)$, $y = h(t)$

$\Rightarrow z$ is a function of t implicitly.

Chain Rule 1

If f, g, h are differentiable, then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

Ex $z = \sqrt{x^2+y^2}$, $x = e^{2t}$, $y = e^{-2t}$

$z = \sqrt{x^2+y^2} = f(x,y)$

Find $\frac{dz}{dt}$

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$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} =$$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} \; ; \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{dx}{dt} = 2e^{2t}, \quad \frac{dy}{dt} = -2e^{-2t}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot 2\underbrace{e^{2t}}_{x} + \frac{y}{\sqrt{x^2+y^2}} \cdot \left(-2\underbrace{e^{-2t}}_{y}\right) = \frac{2x^2 - 2y^2}{\sqrt{x^2+y^2}}$$