University of Idaho

Lecture 21

Chain Rule 2

Now, let $z=f(x,y)$ and $x=g(s,t)$, $y=h(s,t)$

$\Rightarrow$ z is a function of s and t implicitly.

If f, g, h are differentiable, then

$$\boxed{\begin{array}{c} \dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial s} \\ \hline \dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial t} \end{array}}$$

Ex $\quad z = \dfrac{x}{y} = f(x,y) \qquad x = se^{t}, \quad y = 1+se^{-t}$

Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{1}{y}\cdot e^{t} - \frac{x}{y^2}\cdot e^{-t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{1}{y}\cdot se^{t} - \frac{x}{y^2}\left(-se^{-t}\right) = \frac{se^{t}}{y} + \frac{xse^{-t}}{y^2}$$

Implicit Differentiation

(I) Function $y=f(x)$ is given implicitly by equation

$F(x,y)=0$

$\Rightarrow \underbrace{F(x, f(x))}_{\text{function of variable } x}=0$

$y=f(x)$

$F(x,y)=y-f(x)=0$

$y^2-e^{y}\cos(xy)=0$

here y is defined implicitly

$\frac{d}{dx}$: $\quad \frac{d}{dx}0=0=\frac{d}{dx}\left(F(x,f(x))\right) \overset{\text{Chain}}{\underset{\text{Rule 1}}{=}}$

$$= \frac{\partial F}{\partial x}\cdot\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}} \qquad (1) \qquad \text{if } \frac{\partial F}{\partial y}\neq 0$$

Implicit Function Thm 1

Let $y=f(x)$ is implicitly defined by $F(x,y)=0$

If F is defined on a disk containing pt (a,b), where $F(a,b)=0$ and $F_y(a,b)\neq 0$, and F_x, F_y are continuous on this disk, then equation $F(x,y)=0$ defines a function $y=f(x)$ implicitly near pt (a,b), and derivative $\frac{dy}{dx}$ is computed by (1):

$$\boxed{\frac{dy}{dx} = - \frac{\partial F/\partial x}{\partial F/\partial y} = - \frac{F_x}{F_y}} \qquad (1)$$

Ex Find $\frac{dy}{dx}$ using (1) where y is defined by

$$e^y \sin x = x + xy$$

Rewrite as $e^y \sin x - x - xy = 0$

$\Rightarrow \quad F(x,y) = e^y \sin x - x - xy$

$F_x = e^y \cos x - 1 - y \qquad F_y = e^y \sin x - 0 - x$

$$\therefore \quad \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y \cos x - 1 - y}{e^y \sin x - x} = \frac{-e^y \cos x + 1 + y}{e^y \sin x - x}$$

II Now $z = f(x,y)$. We can write this function implicitly as $F(x,y,z) = 0 \iff \underbrace{F(x, y, f(x,y))}_{\text{function of } x \text{ and } y \text{ implicitly}} = 0$

$$\frac{\partial}{\partial x}: \quad 0 = \frac{\partial F}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y} \cdot \cancelto{0}{\frac{\partial y}{\partial x}} + \frac{\partial F}{\partial z} \cdot \underbrace{\frac{\partial z}{\partial x}}$$

x, y are independent variables

$$\therefore \quad \boxed{\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}} \qquad \text{provided } \frac{\partial F}{\partial z} \neq 0$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}}$$

$\frac{\partial F}{\partial z} \neq 0$

Implicit Function Thm 2

$z = f(x,y)$ is definee implicitly by $F(x,y,z) = 0$

If $F(x,y,z)$ is defined on a sphere containing pt (a,b,c), where $F(a,b,c) = 0$, $F_z(a,b,c) \neq 0$, and F_x, F_y, F_z are continuous on the sphere, then eq $F(x,y,z) = 0$ defines $z = f(x,y)$ implicitly and $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are computed by

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{F_x}{F_z} \\ \frac{\partial z}{\partial y} &= -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{F_y}{F_z} \end{aligned}}$$

provided $F_z \neq 0$ (2)

Ex Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using (2) where z is defined implicitly by $e^z = xyz$.

$$z = f(x,y) \iff F(x,y,z) = 0$$

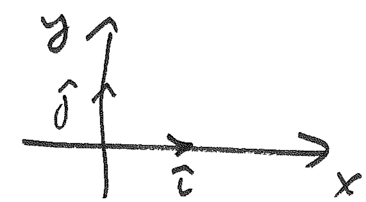
$$F(x,y,z) = e^z - xyz = 0$$

$$F_x = -yz, \quad F_y = -xz, \quad F_z = e^z - xy$$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}$$

14.6 Directional Derivatives and the Gradient Vector



$T = T(x,y)$: temperature

T_x : rate of change of T in positive x-direction, ie. in the direction of unit vector $\hat{i}$

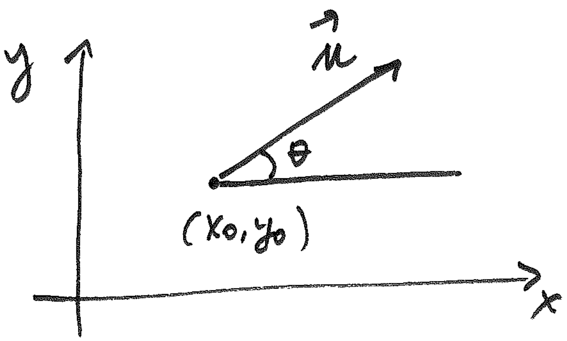
T_y : rate of change of T in positive y-direction, ie. in the direction of unit vector $\hat{j}$

Directional derivative : rate of change of a function in the direction given by some unit vector $\vec{u} = \langle a, b \rangle$.

Consider $z = f(x,y)$. Recall

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$



$\vec{u}$: unit vector given by angle θ

$\vec{u} = \langle \cos\theta, \sin\theta \rangle = \langle a, b \rangle$

Graph of $z = f(x, y)$ is surface S.