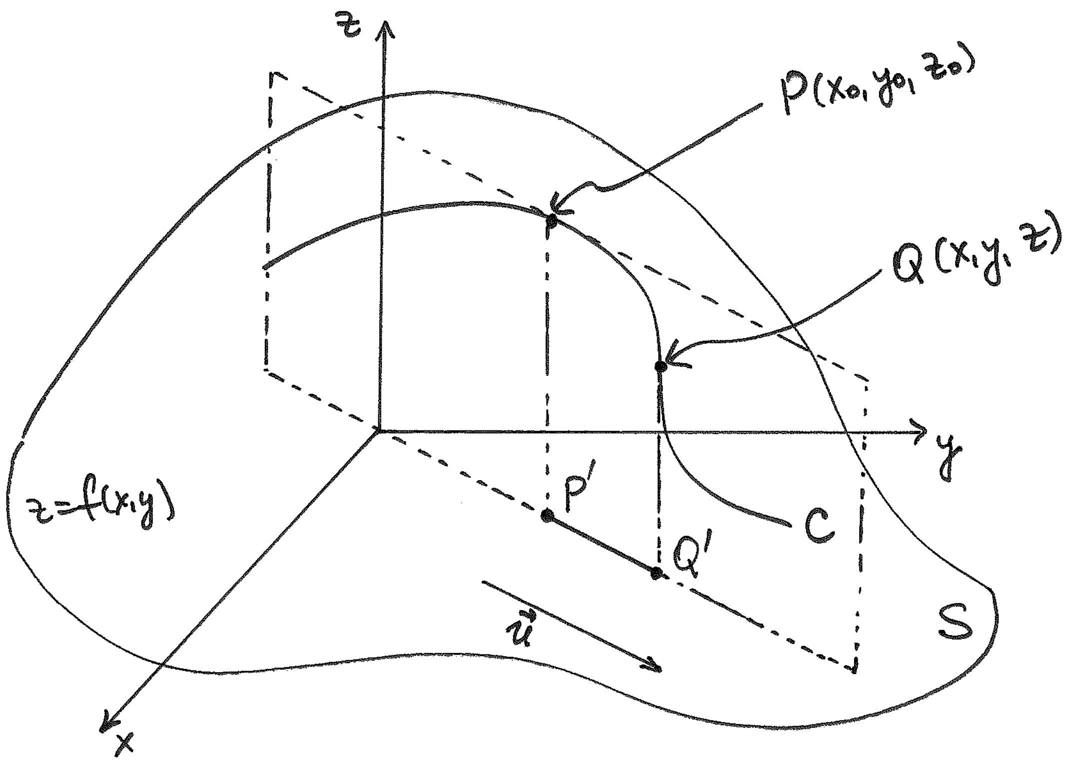
Directional Derivative (Cont'd)



Graph of $z = f(x,y)$ is surface S.

$P(x_0, y_0, z_0)$: point on S

$z_0 = f(x_0, y_0)$

Take a cross-section of this surface with plane through pt P parallel to $\vec{u}$. The intersection is the curve C.

$Q(x, y, z)$ is another pt on C.

P', Q': projections of P, Q onto xy-plane

Let T: tangent line to curve C

$\overrightarrow{P'Q'} \parallel \vec{u} \Rightarrow \overrightarrow{P'Q'} = h\vec{u}$, h: scalar

$\overrightarrow{P'Q'} = \langle x - x_0, y - y_0 \rangle = h \langle a, b \rangle \Rightarrow x - x_0 = ha, \quad y - y_0 = hb$

or $\boxed{x = x_0 + ha, \quad y = y_0 + hb}$

Now, consider

$$\frac{z-z_0}{h} = \frac{\Delta z}{h} = \frac{f(x,y)-f(x_0,y_0)}{h} = \frac{f(x_0+ha, y_0+hb)-f(x_0,y_0)}{h}$$

Take limit as $h \to 0$. This limit is called <u>directional</u> <u>derivative</u> of f in the direction of $\vec{u}$ if this limit exists.

<u>Def</u> The <u>directional derivative</u> of f at (x_0, y_0) in the direction of <u>unit</u> vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb)-f(x_0,y_0)}{h}$$

if limit exists.

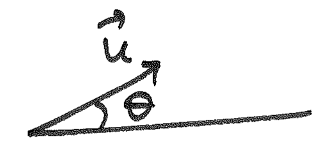
<u>Note</u> if $\vec{u} = \hat{i} = \langle 1, 0 \rangle \Rightarrow D_{\hat{i}} f = f_x$

if $\vec{u} = \hat{j} = \langle 0, 1 \rangle \Rightarrow D_{\hat{j}} f = f_y$

To compute directional derivatives, we use the following result.

Thm If f is a differentiable function of x and y, then f has a directional derivative in the direction of vector $\vec{u} = \langle a, b \rangle$:

$\vec{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$



$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y) \cdot a + f_y(x,y) \cdot b}$$

or

$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y) \cdot \cos\theta + f_y(x,y) \cdot \sin\theta}$$

if $\vec{u}$ is given by angle θ.

Ex Find $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^2y^3 - y^4$ at $(2,1)$ if $\vec{u}$ is a unit vector given by the angle $\theta = \frac{\pi}{6}$.

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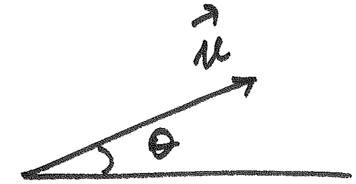
$f_x = 2xy^3\,, \quad f_y = 3x^2y^2 - 4y^3$

$f_x(2,1) = 2\cdot 2\cdot 1^3 = 4\,, \quad f_y(2,1) = 3\cdot 2^2\cdot 1^2 - 4\cdot 1^3 = 8$



$\vec{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$D_{\vec{u}} f(2,1) = f_x(2,1)\cdot\cos\frac{\pi}{6} + f_y(2,1)\cdot\sin\frac{\pi}{6} = 4\cdot\frac{\sqrt{3}}{2} + 8\cdot\frac{1}{2} = \boxed{2\sqrt{3}+4}$

The Gradient Vector

$\vec{u} = \langle a, b \rangle$

Recall

$D_{\vec{u}} f = f_x(x,y)\cdot a + f_y(x,y)\cdot b =$

$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle = \langle f_x, f_y \rangle \cdot \vec{u}$

dot product

<u>Def</u> If f is a function of two variables x and y, the <u>gradient</u> of f is the vector ∇f defined by

$$\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \langle f_x, f_y \rangle = f_x \cdot \vec{i} + f_y \cdot \vec{j}$$

Another notation: $\overrightarrow{\text{grad}}\, f$

Hence,

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

<u>Ex</u> If $f(x,y) = \frac{y^2}{x}$, then

(a) $\nabla f = \langle f_x, f_y \rangle = \langle -\frac{y^2}{x^2}, \frac{2y}{x} \rangle$

(b) $\nabla f \big|_{(1,2)} = \langle -\frac{2^2}{1^2}, \frac{2 \cdot 2}{1} \rangle = \langle -4, 4 \rangle$

(c) Find the rate of change of f at $P(1,2)$ in the direction of $\vec{u} = \frac{1}{3}(2\hat{\imath} + \sqrt{5}\hat{\jmath})$

Check: $|\vec{u}| = \frac{1}{3}\sqrt{2^2 + \sqrt{5}^2} = 1$ ✓

$\Rightarrow \vec{u}$ is a unit vector

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} =$

$= \langle -4, 4 \rangle \cdot \frac{1}{3}\langle 2, \sqrt{5} \rangle = \frac{1}{3}(-4 \cdot 2 + 4 \cdot \sqrt{5}) = \boxed{\frac{4(\sqrt{5}-2)}{3}}$

Functions of Three Variables

Def Directional derivative of $f(x,y,z)$ at (x_0, y_0, z_0) in the direction of unit vector $\vec{u} = \langle a, b, c \rangle$ is

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if the limit exists.

More compact form:

$$D_{\vec{u}}(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

where $\vec{x}_0 = (x_0, y_0, z_0)$ $\quad \vec{u} = \langle a, b, c \rangle$ $\quad$ in 3D

$\vec{x}_0 = (x_0, y_0)$ $\quad \vec{u} = \langle a, b \rangle$ $\quad$ in 2D

Note $\vec{x} = \vec{x}_0 + h\vec{u}$: vector eq^{n} of a line in the direction of $\vec{u}$

$\Rightarrow f(\vec{x}_0 + h\vec{u})$: values of f on that line

If f is differentiable, then

$$D_{\vec{u}} f(x,y,z) = f_x(x,y,z)\cdot a + f_y(x,y,z)\cdot b + f_z(x,y,z)\cdot c =$$

$$= \underbrace{\langle f_x, f_y, f_z \rangle}_{\nabla f} \cdot \underbrace{\langle a, b, c \rangle}_{\vec{u}} = \nabla f \cdot \vec{u}$$

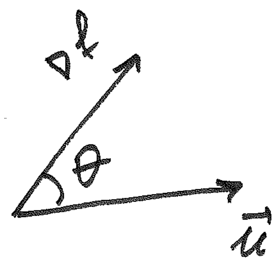
Gradient in 3D:

$$\nabla f = \langle f_x, f_y, f_z \rangle = f_x \cdot \hat{i} + f_y \cdot \hat{j} + f_z \cdot \hat{k}$$

$$\Rightarrow \boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}}$$

Maximizing Directional Derivative

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| \cdot \underbrace{|\vec{u}|}_{=1} \cdot \cos\theta = |\nabla f| \cdot \cos\theta$$



θ: angle between ∇f and $\vec{u}$

max value of $D_{\vec{u}} f$ occurs when $\cos\theta = 1$ or $\theta = 0$.

$\therefore$ $\vec{u}$ and ∇f have the same direction.

Thm Suppose f is differentiable function of two variables. The max value of $D_{\vec{u}} f$ is $|\nabla f(x)|$ and it occurs when $\vec{u}$ and ∇f have the same direction.

Note $D_{\vec{u}} f = 0$ if $\cos\theta = 0$ or $\theta = \frac{\pi}{2}$. $\Rightarrow \vec{u} \perp \nabla f$