Lecture 23

Tangent Planes to Level Surfaces

Consider $F(x, y, z)$: function of three variables x, y, z

$F(x, y, z) = k = \text{const}$: <u>level surface</u> of F, call it S

Let $P(x_0, y_0, z_0)$ be a point on S and let C be any curve that goes through pt P and lies on S.

We can parametrize curve C as

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Let $t = t_0$ correspond to pt P, ie.

$$x_0 = x(t_0), \quad y_0 = y(t_0), \quad z_0 = z(t_0)$$

Since C lies on the surface S $\Rightarrow$

$$F(x(t), y(t), z(t)) = k = \text{const}: \quad \text{function of } t \text{ implicitly}$$

$\frac{d}{dt} \quad \Rightarrow \quad \frac{d}{dt}$ const $= 0$

$$\therefore \quad 0 = \frac{dF}{dt} \underset{\text{rule}}{\overset{\text{chain}}{=}} \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt} =$$

$$= \underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\nabla F} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\vec{r}\,'(t)} = \nabla F \cdot \vec{r}\,'(t)$$

dot product

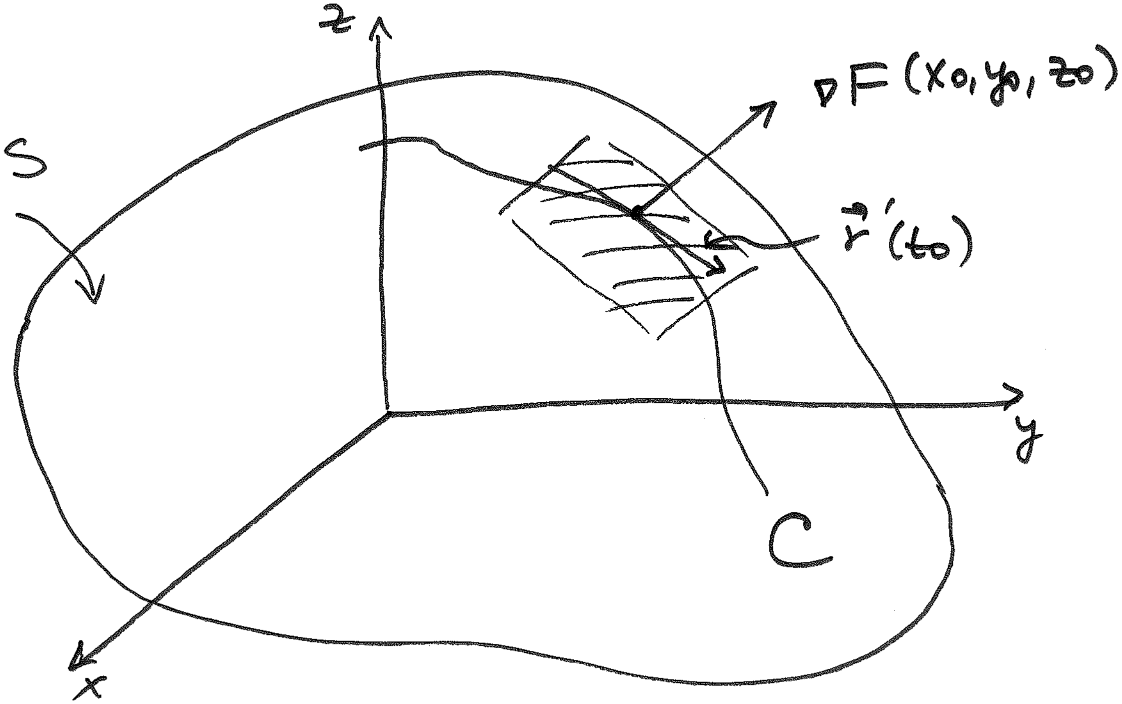
$\Rightarrow \nabla F \cdot \vec{r}\,'(t) = 0$ for any t

In particular, this is also true at $t = t_0$:

$$\Rightarrow \nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$$

$\Rightarrow \nabla F(x_0, y_0, z_0)$ is $\perp$ to the tangent vector $\vec{r}\,'(t_0)$ to <u>any</u> curve C on the level surface S and goes through pt $P(x_0, y_0, z_0)$.

$S: \quad F(x, y, z) = k$



If $\nabla F(x_0, y_0, z_0) \neq 0$, then we can use it as a normal vector to the tangent plane to the level surface S at $P(x_0, y_0, z_0)$:

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0} \quad (*)$$

<u>tangent plane to level ~~sur~~ surface</u>

<u>$S \quad F(x, y, z) = k$ at $P(x_0, y_0, z_0)$</u>

Def Normal line to S at P is a line that goes through pt P and $\perp$ to tangent plane. This line is $\parallel$ to ∇F.

$$\boxed{\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}}$$

normal line to level surface $F(x,y,z)=k$ at $P(x_0,y_0,z_0)$

Special case: S is a surface of the form $z=f(x,y)$

Set $F(x,y,z) = f(x,y) - z$

We can regard F as a level surface with $k=0$, i.e.

$F(x,y,z) = f(x,y) - z = 0.$

Then $F_x = f_x$, $F_y = f_y$, $F_z = -1$

evaluate these partial derivatives at (x_0, y_0, z_0) and substitute into eq$^{\underline{n}}$ (*) of the tangent plane to a level surface:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0 \quad (*)$$

$$\Rightarrow f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

or

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

or

$$z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

This is the eq$^{\underline{n}}$ of the tangent plane to surface $z = f(x,y)$ at (x_0, y_0, z_0).

Summary

* $\nabla f(x,y)$ or $\nabla f(x,y,z)$ gives the direction of fastest increase of f (since $D_{\nabla f} f = (D_{\vec{u}} f)_{max} = |\nabla f|$)

* $-\nabla f(x,y)$ or $-\nabla f(x,y,z)$ gives the direction of fastest decrease of f

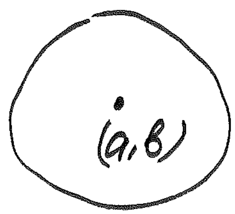
* f does not change in direction $\vec{u} \perp \nabla f$

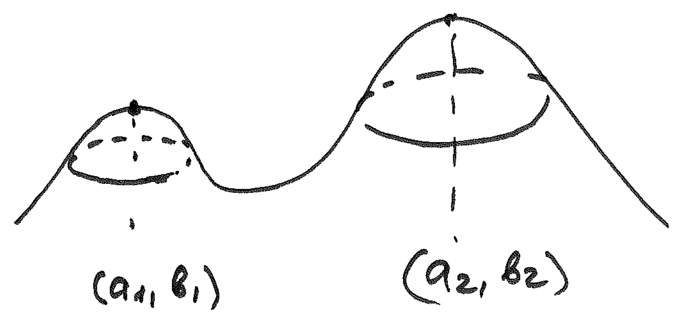
* $\nabla f(x,y)$ is $\perp$ to level curves $f(x,y) = k = const$

* $\nabla f(x,y,z)$ is $\perp$ to level surfaces $f(x,y,z) = k = const$

14.7 Maximum and Minimum Values

$f(x,y)$ has <u>local max</u> at (a,b) if $f(a,b) \geq f(x,y)$ for all $(x,y) \in D$ that are nearby (a,b). The largest value of local max is the <u>global maximum</u>.





Similarly, $f(x,y)$ has <u>local min</u> at (a,b) if $f(a,b) \leq f(x,y)$ for all $(x,y) \in D$ nearby (a,b). The smallest local min is the <u>global minimum</u>.

If the above inequalities are satisfied for all $(x,y) \in D$, we have global maximum or global minimum, respectively.

<u>Thm</u> If f has a local max or local min at (a, b), and if first order partial derivatives f_x and f_y exist, then

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0$$

a, b

$z_0 = f(a,b)$

Recall

$$z = z_0 + \cancelto{0}{f_x(a,b)}(x - x_0) + \cancelto{0}{f_y(a,b)}(y - y_0):$$ tangent plane to $z = f(x,y)$ at (a, b, z_0)

$\Rightarrow \quad z = z_0$

$\therefore$ if f has a local max or local min at (a, b), then the tangent plane is <u>horizontal</u> at (a, b, z_0).

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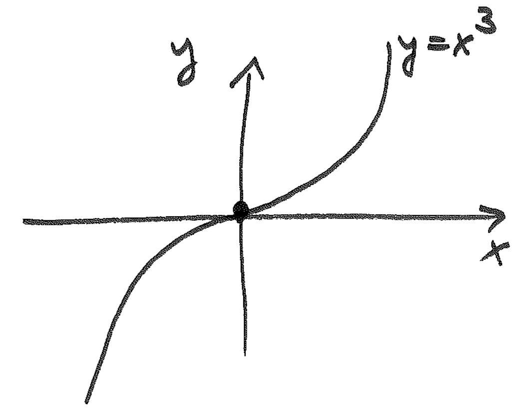
Def A pt (a,b) is a <u>critical point</u> / <u>stationary pt</u> of f if

$f_x(a,b)=0$ and $f_y(a,b)=0$

or one of them does not exist.



Ex $y=x^3$

$y'=3x^2=0$ at $x=0$ $\Rightarrow$ $x=0$ is a crit. pt

but $x=0$ is not a pt of local min or local max

f has a local max or local min at (a,b) $\Rightarrow$, $\not\Leftarrow$ pt (a,b) is a critical point

Ex $f(x,y) = x^2+y^2-2x-6y+14$

$f_x = 2x-2 = 2(x-1)$, $f_y = 2y-6 = 2(y-3)$

$$f_x = f_y = 0 \Rightarrow \left.\begin{array}{l} 2(x-1)=0 \Rightarrow x=1 \\ 2(y-3)=0 \Rightarrow y=3 \end{array}\right\} \Rightarrow (1,3) \text{ is a crit. pt}$$

$f(1,3) = 1^2+3^2-2\cdot 1-6\cdot 3+14 = 4$