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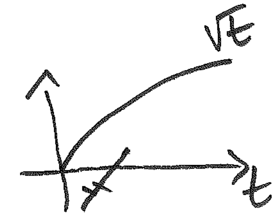
Lecture 24

Review 2

#19
S 13.4

The position function of a particle is given by $\vec{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?



$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, 5, 2t-16 \rangle$

velocity

speed $v = |\vec{v}(t)| = \sqrt{4t^2 + 25 + (2t-16)^2} \longrightarrow$ min ?

v has min at some time t^* when v^2 has min at t^*

$\sqrt{t}$: monotonically increasing & continuous function

Consider

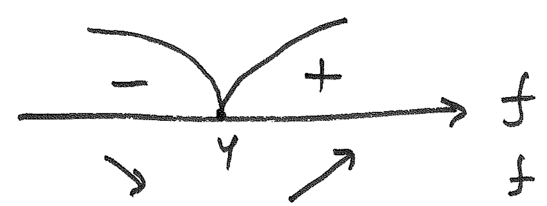
$$f(t) = 4t^2 + 25 + (2t-16)^2$$

$$f'(t) = 8t + 2(2t-16)\cdot 2 = 8[t + t - 8] = 16(t-4)$$

$f'(t) = 0$ when $t = 4$

$8[2t-8]$

$8\cdot 2[t-4]$



$f(t)$ achieves its min at $t=4$ $\Rightarrow$ $v_{min} = v(4) = \sqrt{4\cdot 4^2+25+(2\cdot 4-16)^2}$

$= \sqrt{4\cdot 4^2+25+64} = \sqrt{153}$

$\Rightarrow$ $\boxed{v_{min} = \sqrt{153}}$ $\boxed{\text{at } t=4}$

<u>Ex</u> Find the limit, if the limit exists, or show that the limit does not exist. Justify your answer.

$f(x,y)$ is continuous at (a,b) $\overset{def}{\Rightarrow}$ $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

(a) $\lim\limits_{(x,y)\to(1,1)} \dfrac{6xy}{3x^2+y^2}$

$\dfrac{6xy}{3x^2+y^2}$ is a rational function and denominator $(3x^2+y^2)\Big|_{(1,1)} \neq 0$

$\Rightarrow$ $\dfrac{6xy}{3x^2+y^2}$ is continuous at $(1,1)$ $\Rightarrow$

$$\lim_{(x,y)\to(1,1)} \frac{6xy}{3x^2+y^2} = \left(\frac{6xy}{3x^2+y^2}\right)\Bigg|_{(1,1)} = \frac{6\cdot 1\cdot 1}{3\cdot 1^2+1} = \frac{6}{4} = \boxed{\frac{3}{2}}$$

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(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{6xy}{3x^2+y^2}$

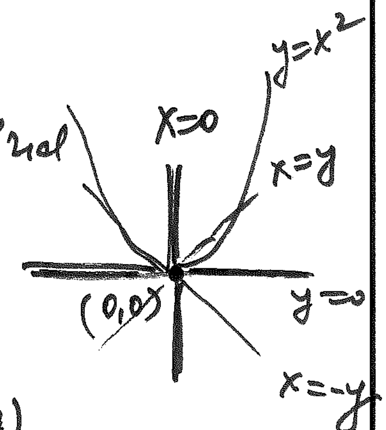
<u>Path 1</u>: $x=y \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{6xy}{3x^2+y^2} = \lim_{x\to 0} \frac{6x^2}{3x^2+x^2} = \lim_{x\to 0} \frac{6\cancel{x^2}}{4\cancel{x^2}} =$

$= \displaystyle\lim_{x\to 0} \frac{6}{4} = \frac{3}{2} = L_1$

<u>Path 2</u>: $x=-y \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{6xy}{3x^2+y^2} = \lim_{x\to 0} \frac{-6x^2}{3x^2+x^2} = -\frac{3}{2} = L_2$

$L_1 \neq L_2 \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{6xy}{3x^2+y^2}$ does not exist

<u>Note</u>: if $L_1 = L_2$, the test is inconclusive $\rightarrow$ need to find other direction along which limit is different



Possible choices: $x=0$ (y-axis) $\quad y=x^2$ (parabola)

$y=0$ (x-axis) $\quad y=x^3$ (cubic parabola)

Path 3:

$x=0$

$$\lim_{(x,y)\to(0,0)} \frac{6xy}{3x^2+y^2} = \lim_{\substack{y\to 0\\ y\neq 0}} \frac{6\cdot 0\cdot y}{3\cdot 0^2+y^2} = \lim_{y\to 0} \underbrace{\frac{0}{y^2}}_{=0} = 0 = L_3 \neq L_1$$

Ex Find partial derivatives

(a) $f(x,y,z) = xyz^2 + (x+e^x)(\sin x + y)$. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

x,y,z: independent variables

$$\frac{\partial f}{\partial x} = yz^2 + \underbrace{(1+e^x)}_{(x+e^x)_x}(\sin x + y) + (x+e^x)\cdot \underbrace{\cos x}_{(\sin x+y)_x}$$

$$\frac{\partial f}{\partial y} = xz^2 + (x+e^x)\cdot 1$$

(b) $z = z(x,y)$ is given implicitly by $xy + yz^3 = x^2 + y + z$

Find $\frac{\partial z}{\partial x}$

x,y: independent variables

z: dependent variable

Let $F(x,y,z) = xy + yz^3 - x^2 - y - z = 0$

Then

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{y - 2x}{3yz^2 - 1}$$

(c) $z = x^2 - y^3$, $x = r\cos t$, $y = r\sin t$

Find $\frac{\partial z}{\partial t}$ without expressing z as a function of r and t explicitly.

$z = z(x,y)$ $\quad x = x(r,t)$, $y = y(r,t)$

We can see that z is a function of r and t implicitly.

$$\frac{\partial z}{\partial t} \underset{\text{rule}}{\overset{\text{chain}}{=}} \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = 2x\cdot(-r\sin t) + (-3y^2)\cdot(r\cos t)$$

$$= -2xr\sin t - 3y^2 r\cos t$$

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Ex Find the equation of a tangent plane to the surface

$z = x^2 + 4y^2$ at $(2, 1, 8)$.

$$z = z_0 + \underbrace{f_x(x_0, y_0)}_{\text{const}}(x - x_0) + \underbrace{f_y(x_0, y_0)}_{\text{const}}(y - y_0) \qquad z_0 = f(x_0, y_0)$$

not a function

$z = f(x,y) = x^2 + 4y^2 \qquad f_x = 2x, \quad f_y = 8y$

$x_0 = 2, \quad y_0 = 1, \quad z_0 = 8$

$f_x(x_0, y_0) = f_x(2,1) = 2 \cdot 2 = 4; \quad f_y(x_0, y_0) = f_y(2,1) = 8 \cdot 1 = 8$

$\therefore \quad z = 8 + 4(x - 2) + 8(y - 1)$

$z = \cancel{8} + 4x - \cancel{8} + 8y - 8$

$\boxed{z = 4x + 8y - 8}$ or $\boxed{4x + 8y - z - 8 = 0}$

Ex Find the linear approximation of $f(x,y) = 1 - xy\cdot\cos(\pi y)$ at $(1,1)$ and use it to approximate $f(1.02, 0.97)$.

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) \approx L(x,y)$$

$x_0 = 1,\ y_0 = 1$ $\qquad f(x_0,y_0) = f(1,1) = 1 - 1\cdot 1\cdot \underbrace{\cos(\pi)}_{-1} = 2$

$f_x = -y\cos(\pi y)$ $\qquad f_x(1,1) = -1\cdot\cos\pi = 1$

$$f_y = -x\left[1\cdot\cos(\pi y) + y\cdot(-\sin(\pi y))\cdot\pi\right]$$

$$f_y(1,1) = -1\left[\underbrace{\cos\pi}_{-1} + 1\cdot\pi\,(-\overset{0}{\cancel{\sin\pi}})\right] = 1$$

$\therefore\ L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$L(x,y) = 2 + 1\cdot(x-1) + 1(y-1) = \boxed{x+y = L(x,y)}$ good approx near $(1,1)$

$$\therefore\ f(1.02, 0.97) \approx L(1.02, 0.97) = 1.02 + 0.97 = \boxed{1.99}$$

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$$|\text{abs. error}| = |\underbrace{f(1.02, 0.97)}_{\text{exact}} - \underbrace{L(1.02, 0.97)}_{\text{approximation}}| =$$

$$|\text{rel. error}| = \frac{\text{abs. error}}{|\text{exact value}|} = \frac{\text{abs. error}}{|f(1.02, 0.97)|}$$

$$\text{perc. error} = |\text{rel. error}| \cdot 100\%$$