Lecture 25

Ex $f(x,y) = x^2 + y^2 - 2x - 6y + 14$ (Cont'd)

We found

$f_x = 2(x-1)$ $\qquad$ $f_y = 2(y-3)$

$f_x = 0, \quad f_y = 0 \quad \Rightarrow \quad (1,3)$ is a critical pt of f

$f(1,3) = 4$

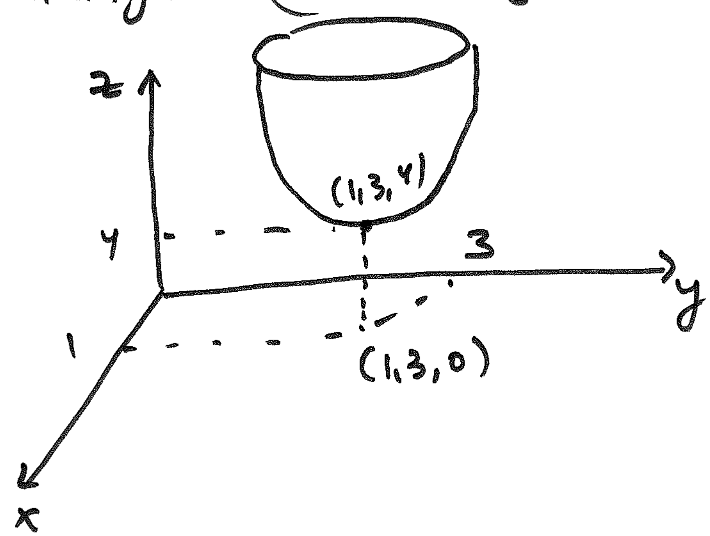
$$f(x,y) = x^2 + y^2 - 2x - 6y + 14 \underset{\text{square}}{\overset{\text{complete}}{=}} (x^2 - 2x + 1) - 1 + (y^2 - 6y + 9)$$

$$-9 + 14 = \underbrace{(x-1)^2}_{\geq 0} + \underbrace{(y-3)^2}_{\geq 0} + 4 \geq 4 = f(1,3)$$

$\Rightarrow f(x,y) \geq f(1,3)$ for all (x,y)

$\therefore$ $(1,3)$ is a point of local and global minimum

$z = f(x,y) = (x-1)^2 + (y-3)^2 + 4$: elliptic paraboloid



<u>Ex</u> $f(x,y) = y^2 - x^2$

$f_x = -2x, \quad f_y = 2y$

$f_x = 0, \; f_y = 0 \Rightarrow \left.\begin{array}{l} -2x = 0 \\ 2y = 0 \end{array}\right\} \Rightarrow (0,0)$ is a critical pt

$f(0,0) = 0$

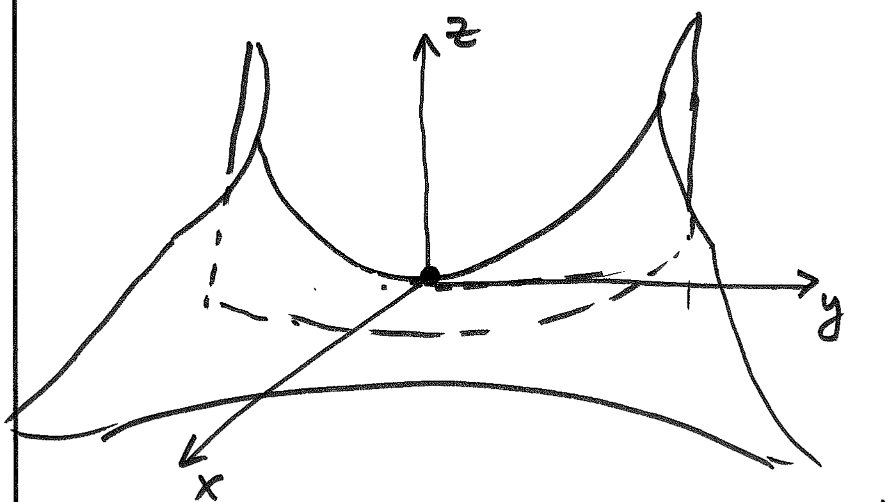
on x-axis: $y = 0 \Rightarrow f(x,y) = -x^2 \leq 0 = f(0,0)$

University of Idaho

on y-axis : $x=0 \Rightarrow f(x,y)=y^2 \geq 0 = f(0,0)$

$\therefore$ f has no extremum at $(0,0)$

$z = f(x,y) = y^2 - x^2$



hyperbolic paraboloid

Second Derivative Test

Suppose 2^{nd} partial derivatives of f are continuous on a disk containing (a,b), and $f_x(a,b)=0$ and $f_y(a,b)$ (i.e. (a,b) is a critical point). Let

$$D = D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} =$$

$$= f_{xx}(a,b) \cdot f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

(a) if $D>0$ and $f_{xx}(a,b)>0$, then f has a local min at (a,b);

(b) if $D>0$ and $f_{xx}(a,b)<0$, then f has a local max at (a,b);

(c) if $D<0$, then (a,b) is not a point of local min or local max. Such point is called a <u>saddle point</u>;

(d) if $D=0 \Rightarrow$ inconclusive test

Ex $f(x,y) = y^2 - x^2$

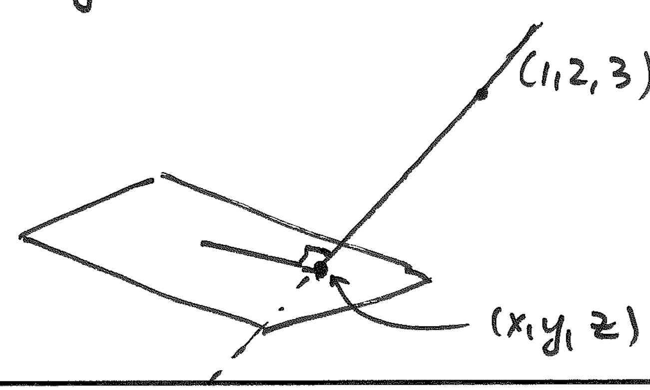
$f_x = -2x, \quad f_y = 2y \Rightarrow (0,0)$ is a crit. pt

$f_{xx} = -2, \quad f_{xy} = 0, \quad f_{yy} = 2$

$$D = D(0,0) = \begin{vmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 < 0$$

$\therefore$ $(0,0)$ is a saddle point

Ex Find the point on the plane $x - y + z = 4$ that is closest to point $(1,2,3)$.



Let (x, y, z) be a pt on the plane $x - y + z = 4$

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$$: distance between $(1,2,3)$ and (x,y,z)

$$d \xrightarrow{?} \min$$

$(x,y,z) \in$ plane $\Rightarrow$ (x,y,z) satisfies the equation $x - y + z = 4$

$\Rightarrow z = 4 - x + y$

$$\therefore \quad d = \sqrt{(x-1)^2 + (y-2)^2 + (4-x+y-3)^2}$$

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (y-x+1)^2} \longrightarrow \min$$

Instead of d, we can consider d^2 and find its min:

$y = \sqrt{x}$ is a monotonically increasing contin. f^{2}

$$f(x,y) \equiv d^2 = (x-1)^2 + (y-2)^2 + (y-x+1)^2 \longrightarrow \min$$

$$f_x = 2(x-1) + 2(y-x+1)(-1) = 2[x-1-y+x-1] = 2[2x-y-2]$$

$$f_y = 2(y-2) + 2(y-x+1) = 2[2y-x-1]$$

$$f_x = f_y = 0 \Rightarrow \begin{cases} 2x-y-2=0 \\ 2y-x-1=0 \end{cases} \Rightarrow x = \frac{5}{3},\ y = \frac{4}{3}$$

$\therefore \left(\frac{5}{3}, \frac{4}{3}\right)$ is a crit. point

$$f_{xx} = 2 \cdot 2 = 4, \quad f_{xy} = -2, \quad f_{yy} = 2 \cdot 2 = 4$$

$$D\Big|_{\left(\frac{5}{3}, \frac{4}{3}\right)} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{\left(\frac{5}{3}, \frac{4}{3}\right)} = \begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} = 16 - 4 = 12 > 0$$

University of Idaho

$D\left(\frac{5}{3}, \frac{4}{3}\right) = 12 > 0$, $f_{xx}\left(\frac{5}{3}, \frac{4}{3}\right) = 4 > 0 \Rightarrow \left(\frac{5}{3}, \frac{4}{3}\right)$ is a pt of local min by Second Derivative test

$$d_{min} = d\left(\frac{5}{3}, \frac{4}{3}\right) = \sqrt{(x-1)^2 + (y-2)^2 + (y-x+1)^2}\,\Big|_{\left(\frac{5}{3}, \frac{4}{3}\right)} = 1.1547$$

To find the point on the plane $x-y+z=4$ that is closest to pt $(1,2,3)$, we need to find z-coordinate. We can use eq^{n} of the plane $x-y-z=4$ and solve for z:

$$z = 4 - x + y$$

$$z\Big|_{\left(\frac{5}{3}, \frac{4}{3}\right)} = 4 - \frac{5}{3} + \frac{4}{3} = \frac{11}{3}$$

$\therefore \left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$: pt closest to $(1,2,3)$