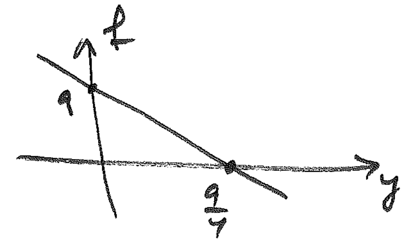
$L_2: \ x=3, \ 0 \le y \le 2 \ \Rightarrow \ f = 3^2 - 2\cdot 3y + 2y = 9 - 4y$

$f_{max} = (9-4y)\big|_{y=0} = \boxed{9}$

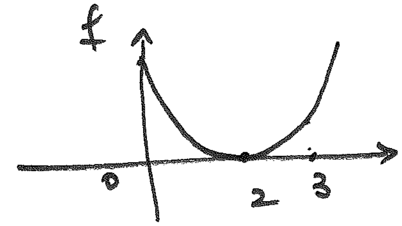
$f_{min} = (9-4y)\big|_{y=2} = \boxed{1}$



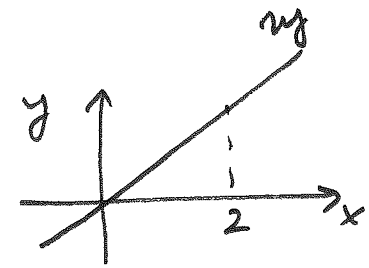
$L_3: \ y=2, \ 0 \le x \le 3 \ \Rightarrow \ f = x^2 - 2x\cdot 2 + 2\cdot 2 = x^2 - 4x + 4 = (x-2)^2$

$f_{min} = (x-2)^2\big|_{x=2} = \boxed{0}$

$f_{max} = (x-2)^2\big|_{x=0} = \boxed{4}$



$L_4: \ x=0, \ 0 \le y \le 2 \ \Rightarrow \ f = 0^2 - 2\cdot 0\cdot y + 2y = 2y$

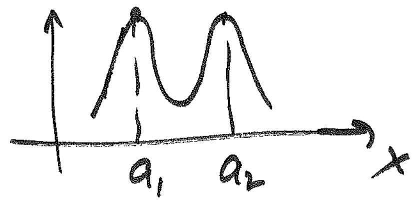


$f_{min} = 2y\big|_{y=0} = \boxed{0}$ $\qquad f_{max} = (2y)\big|_{y=2} = \boxed{4}$

$\therefore \ \boxed{f_{abs\,max} = 9 = f(3,0) \qquad f_{abs\,min} = 0 = f(0,0) = f(2,2)}$

University of Idaho

In 1D

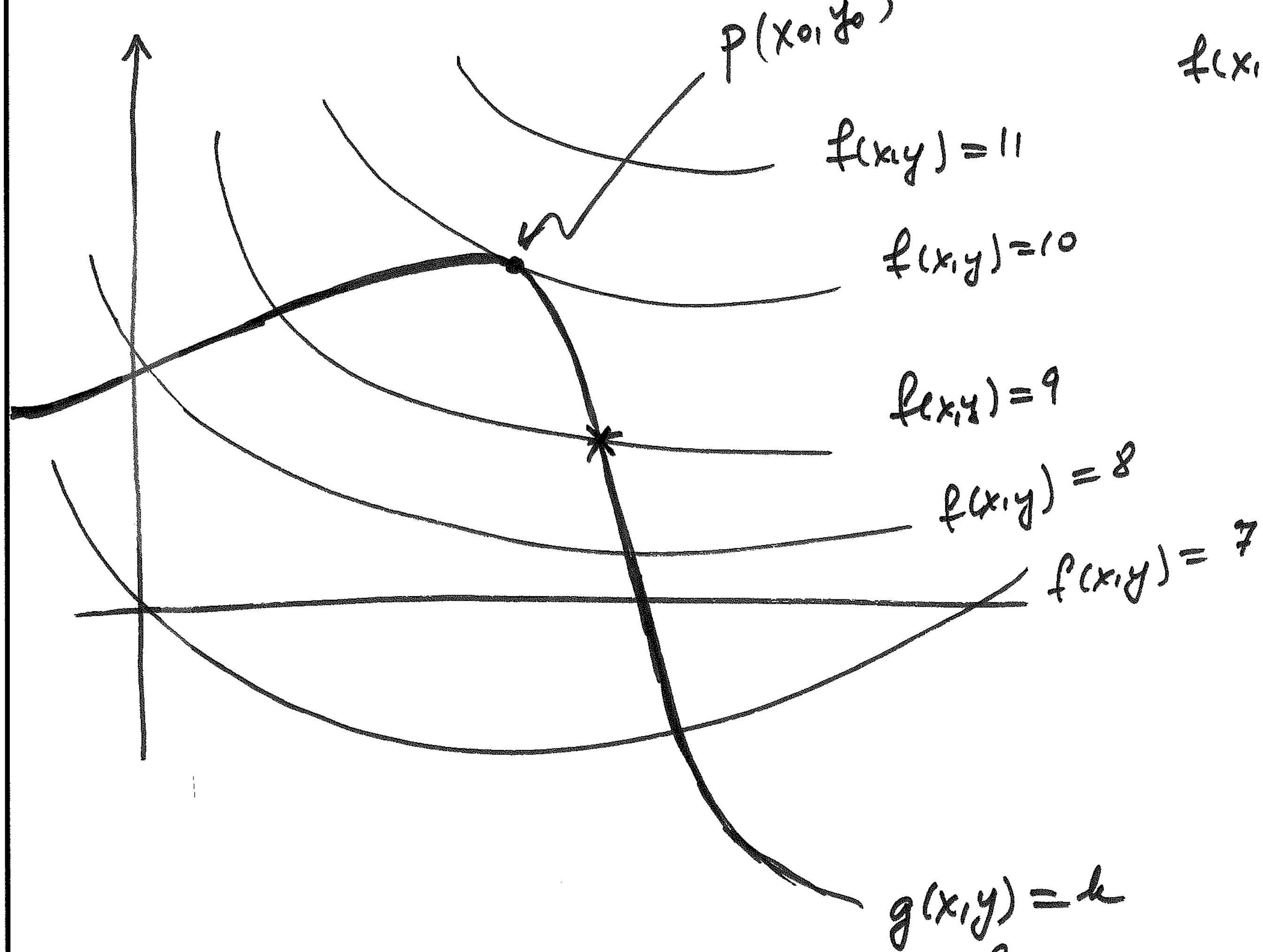


abs. max or abs min can occur at more than one point

14.8 Method of Lagrange multipliers

Goal: to find max or min of $f(x,y,z)$ subject to a constraint $g(x,y,z)=k$

We will consider 2D case first: find max or min of $f(x,y)$ subject to a constraint given by a level curve $g(x,y)=k$, i.e. find a pt (x,y) at which f attains its max or min value such that pt (x,y) stays on the level curve $g(x,y)=k$.



$f(x,y) = c$: level curves

<u>Goal</u>: to maximize/minimize $f(x,y)=c$ subject to the constraint $g(x,y)=k$

From the graph, the max value of f occurs at a
point, call it $P(x_0, y_0)$, at which level curves
$f(x,y) = c$, $g(x,y) = k$ touch each other. At any
other point of their intersection, we can still move
along level curve $g(x,y) = k$ to increase or decrease
values of f. At the point P of intersection (level
curves touch each other), level curves have the same
tangent line $\Rightarrow$ they have the same normal line.
From Section 14.6, we know that normal line is $\parallel$
to gradient of a level curve.

$\Rightarrow$ at $P(x_0, y_0)$ $\quad \nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$

$$\left.\begin{aligned} &\Rightarrow \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \\ &+ \text{constraint} \quad g(x_0, y_0) = k \end{aligned}\right\} \text{system of 3 equations for 3 unknowns } x_0, y_0, \lambda$$

λ: <u>Lagrange multiplier</u>

Method of Lagrange multipliers for $f(x,y)$ (2D case)

To find values of $f(x,y)$ subject to a constraint $g(x,y) = k$:

(a) Find all values x, y, λ:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

subject to

$$g(x,y) = k$$

(b) Evaluate f at (x,y) found in (a). The largest is f_{max}, the smallest f_{min}.

Recall, $\nabla f = \langle f_x, f_y \rangle$, $\nabla g = \langle g_x, g_y \rangle$

In component form

$$\begin{cases} f_x = \lambda g_x, \quad f_y = \lambda g_x \\ g(x,y) = k \end{cases}$$

system of 3 equations for 3 unknowns x, y, λ

University of Idaho

Back to 3D

$f(x, y, z)$

$f(x, y, z) = c$: level surfaces

$g(x, y, z) = k$: level surface

<u>Goal</u>: to find extreme values of $f(x, y, z)$ subject to a constraint given by a level surface $g(x, y, z) = k$, i.e. find a point (x, y, z) at which f has max or min, such that (x, y, z) remains on the level surface $g(x, y, z) = k$.

Similarly to 2D, f attains its max or min value at points where level surfaces $f(x, y, z) = c$ and $g(x, y, z) = k$ touch each other.

Let f have max at $P(x_0, y_0, z_0)$ $\Rightarrow$ level surfaces share the same tangent plane at (x_0, y_0, z_0) and the same normal line whose direction is given by $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$.

$\Rightarrow \nabla f(x_0, y_0, z_0) \parallel \nabla g(x_0, y_0, z_0)$

$\Rightarrow \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

To close the system, we add the constraint

$$g(x_0, y_0, z_0) = k$$

System of 4 equations for 4 unknowns x_0, y_0, z_0 and λ

Method of Lagrange multipliers for $f(x,y,z)$ 3D case

To find max or min values of $f(x,y,z)$ subject to $g(x,y,z) = k$ $(\nabla g(x,y,z) \neq \vec{0})$:

(a) Find all values of x, y, z and λ for which

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

subject to

$$g(x,y,z) = k$$

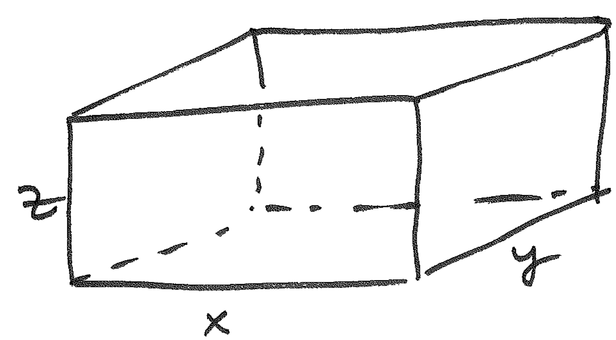
(b) Evaluate f at all points (x, y, z) found in (a). The largest is f_{max}, the smallest is f_{min}.

Note component form:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x, y, z) = k \end{cases}$$

system of 4 equations for 4 unknowns x, y, z, λ

Ex A rectangular box without a lid is to be made from $12\ m^2$ of cardboard. Find max volume of such a box.



Solution

$V = xyz \longrightarrow \max$ ← Volume of a box

$V = f(x, y, z)$

University of Idaho

Constraint: surface area

$$g(x,y,z) = xy + 2xz + 2yz = 12 \ (m^2)$$

We will use the method of Lagrange multipliers

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = k \end{cases}$$

or in component form

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = k \end{cases}$$

$$\nabla f = \langle yz, xz, xy \rangle \qquad \nabla g = \langle y+2z, x+2z, 2x+2y \rangle$$

University of Idaho

Hence

$$\begin{cases} yz = \lambda(y+2z) & (1) \quad \Big|\cdot x \\ xz = \lambda(x+2z) & (2) \quad \Big|\cdot y \\ xy = \lambda(2x+2y) & (3) \quad \Big|\cdot z \\ xy + 2xz + 2yz = 12 & (4) \end{cases}$$

system of 4 nonlinear equations for x, y, z and λ

(1') $xyz = \lambda x(y+2z)$

(2') $xyz = \lambda y(x+2z)$

(3') $xyz = \lambda z(2x+2y)$

From (1'), (2') $\Rightarrow$ $\lambda x(y+2z) = \lambda y(x+2z) \quad \Big| \frac{1}{\lambda}$ (5)

$\lambda \neq 0$

Check if $\lambda = 0$ is a solution.

Let $\lambda = 0$ be a solution

$\Rightarrow$ from (1) $\Rightarrow$ $yz = 0$

from (2) $\Rightarrow$ $xz = 0$

from (3) $\Rightarrow$ $xy = 0$

Constraint: $\underbrace{xy}_{=0} + 2\underbrace{xz}_{=0} + 2\underbrace{yz}_{=0} = 12 \Rightarrow 0 = 12$ ↯

$\therefore$ $\lambda = 0$ is not a solution

(5) $\Rightarrow$

$$x(y+2z) = y(x+2z)$$

$$\cancel{xy} + 2xz = \cancel{xy} + 2yz$$

$$xz = yz \quad \Big| \ \frac{1}{z} \qquad z \neq 0: \text{ height}$$

$$\Rightarrow \boxed{x = y}$$

similarly, from (2'), (3')

University of Idaho

$$\lambda y(x+2z) = \lambda z(2x+2y) \quad \Big| \ \frac{1}{\lambda} \qquad \lambda \neq 0$$

$$y(x+2z) = z(2x+2y)$$

$$yx + \cancel{2yz} = 2xz + \cancel{2yz}$$

$$yx = 2xz \quad \Big| \ \frac{1}{x} \qquad x \neq 0 : \text{length}$$

$$\boxed{y = 2z}$$

Now we can use the constraint

$$\underset{=y}{x}y + \overset{=y}{2x}\underbrace{z}_{y} + \overset{=y}{2yz} = 12$$

$$y^2 + y^2 + y^2 = 12 \Rightarrow 3y^2 = 12 \Rightarrow y^2 = 4 \Rightarrow \boxed{y = 2}$$

$y = \pm 2$ but y is length $\rightarrow y > 0$

$$\Rightarrow z = \frac{1}{2}y = 1 \qquad x = y = 2$$

$\therefore \quad x = 2 \quad y = 2 \quad z = 1$

$$V_{max} = V\Big|_{\substack{x=2\\ y=2\\ z=1}} = 2 \cdot 2 \cdot 1 = 4 \ (m^3)$$