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Lecture 28

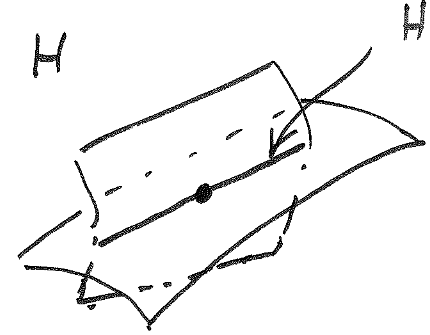
Method of Lagrange multipliers: Two Constraints

$f(x,y,z) \to$ min or max

subject to two constraints

$g(x,y,z)=k$ and $h(x,y,z)=c$

Geometrically: look for extreme values of f where (x,y,z) is restricted to lie on the curve H of intersection of two level surfaces $g(x,y,z)=k$ and $h(x,y,z)=c$.



Let $P(x_0,y_0,z_0)$ be a point where f attains max or min $\Rightarrow$

$$\nabla f \perp H$$

(from the discussion of the case w/ one constraint)

At the same time, $\nabla g \perp$ level surface $g(x,y,z)=k$

and $\nabla h \perp$ level surface $h(x,y,z)=c$

$\therefore \nabla g \perp H$ (since H is the intersection of level surfaces $g(x,y,z)=k$ and $h(x,y,z)=c$

and $\nabla h \perp H$

Hence, $\nabla f(x_0,y_0,z_0)$ is in the plane determined by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$, i.e.

$$\left.\begin{aligned} \nabla f(x_0,y_0,z_0) &= \lambda \nabla g(x_0,y_0,z_0) + \mu \nabla h(x_0,y_0,z_0) \\ \text{+ 2 constraints:}\quad g(x_0,y_0,z_0) &= k \\ h(x_0,y_0,z_0) &= c \end{aligned}\right\} \begin{array}{l} \text{system of 5} \\ \text{eq}^{\text{ns}}\text{ for} \\ \text{5 unknowns} \\ x_0, y_0, z_0, \lambda, \mu \end{array}$$

Problem: look for extreme values of $f(x,y,z)$ subject to two constraints:

$g(x,y,z) = k$ and $h(x,y,z) = c$

Method of Lagrange multipliers: Find all values x, y, z, λ, μ such that

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x,y,z) = k \\ h(x,y,z) = c \end{cases}$$

: vector equation

or in component form:

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x,y,z) = k \\ h(x,y,z) = c \end{cases}$$

5 eq^{ns} for 5 unknowns x, y, z, λ, μ

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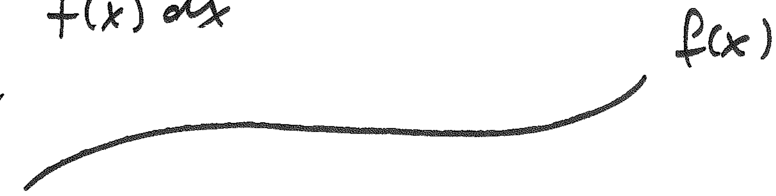
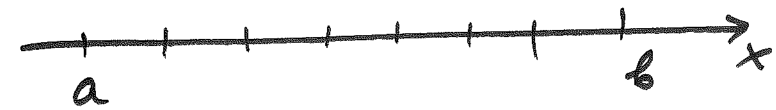
Chapter 15 Multiple Integrals

15.1 Double Integrals over Rectangles

Review of definite integrals: $\int_a^b f(x)\,dx$



$f(x), \quad x \in [a, b]$



n subintervals $[x_{i-1}, x_i]$

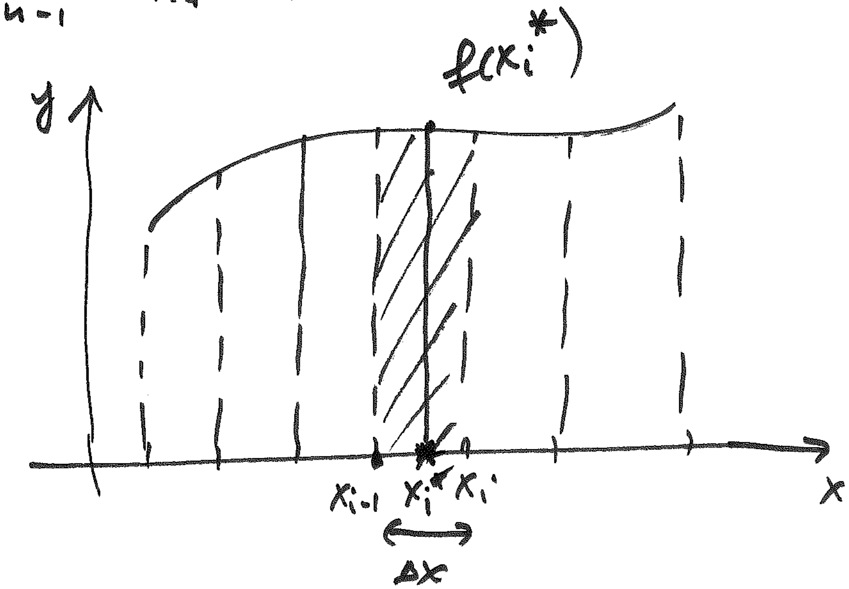
w/ length $\Delta x = \frac{b-a}{n}$

Partition: $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$

let $f(x) \geq 0$

x_i^*: sample point

$f(x_i^*)\Delta x$: area of one subrectangle



$\sum_{i=1}^{n} f(x_i^*)\Delta x \approx$ area under function $f(x)$ above x-axis

Riemann sum

$\int_a^b f(x)\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ if limit exist

definite integral of f over $[a, b]$

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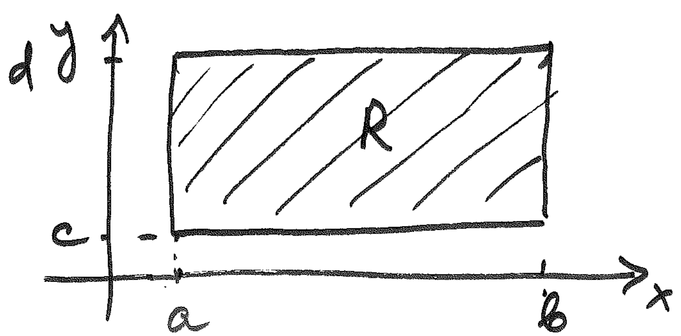
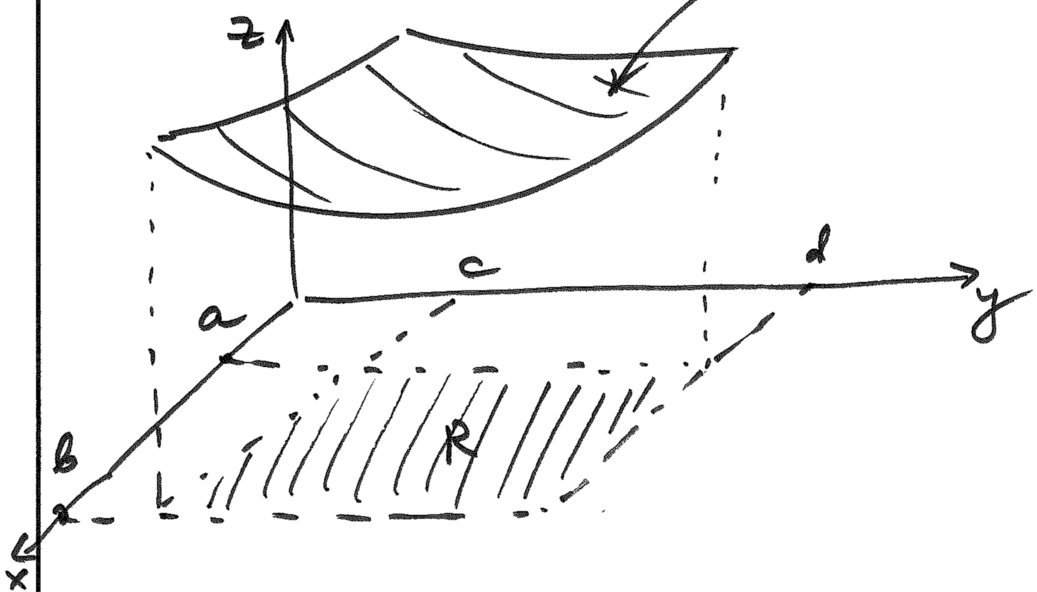
Volumes and Double Integrals

$f(x,y)$ $\qquad R = \{(x,y):\ a \le x \le b,\ \ c \le y \le d\} = [a,b]\times[c,d]$

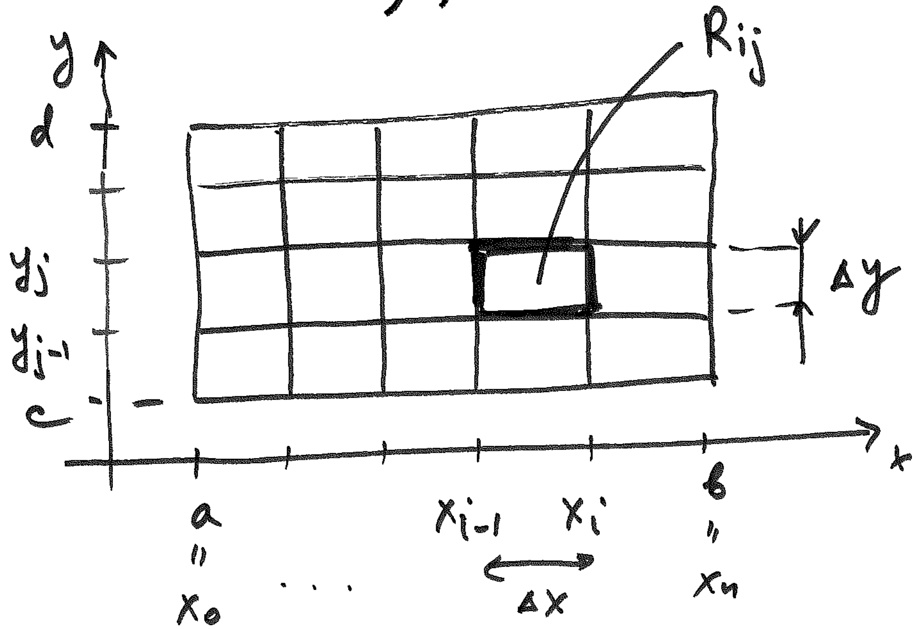


Suppose $f(x,y) \geq 0$

$z = f(x,y)$: surface



Goal: to compute volume of the region bounded by surface $z = f(x,y)$ and rectangle R.



Divide $[a, b]$ into n subintervals of length

$$\Delta x = \frac{b-a}{n}$$

$[c, d]$ into m subintervals of length $\Delta y = \dfrac{d-c}{m}$

Partitions: $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$

$c = y_0 < y_1 < \ldots < y_{m-1} < y_m = d$

Area of R_{ij} is $\Delta x \cdot \Delta y = \Delta A$

(x_{ij}^*, y_{ij}^*): sample point in R_{ij}

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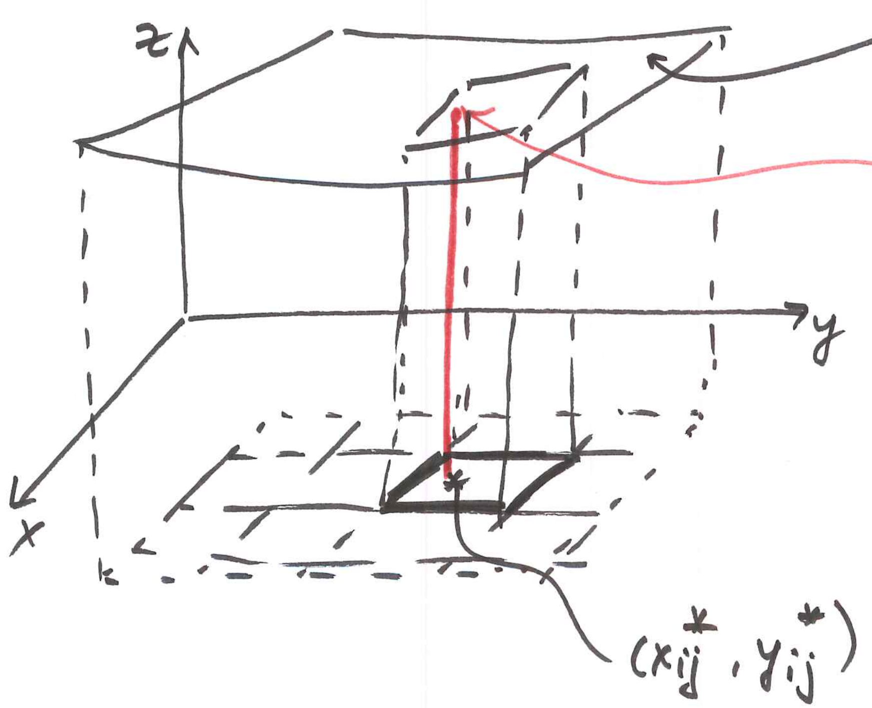

Volume of this small column is

$\approx \Delta A \cdot f(x_{ij}^*, y_{ij}^*)$

$V = \text{Volume}_{\text{solid}} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A$

<u>double Riemann sum</u>

Define

$$V = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A \; :$$

volume of a solid bounded by surface $z = f(x,y)$ and rectangle R

if limit exists.

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<u>Def</u> a <u>double integral</u> of $f(x,y)$ over rectangle R is defined

$$\iint_R f(x,y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

if limit exists.

<u>Note</u>: if $f(x,y)$ is continuous $\Rightarrow$ f is integrable, i.e. $\iint_R f$ is a finite #.

<u>Note</u> if $f \geq 0$ $\Rightarrow$ $\iint_R f\, dA$ is the volume of the solid bounded by surface $z = f(x,y)$ and rectangle R

$\Rightarrow$ $\boxed{\iint_R f\, dA = V}$ $\boxed{f \geq 0}$

Midpoint Rule

Sample points are centers of R_{ij}:

$$x_{i-1} \le x \le x_i \qquad y_{j-1} \le y \le y_j$$

$$\Rightarrow \quad \bar{x}_i = \frac{x_{i-1} + x_i}{2} \qquad \bar{y}_j = \frac{y_{j-1} + y_j}{2}$$

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$$\Delta A = \Delta x \cdot \Delta y$$