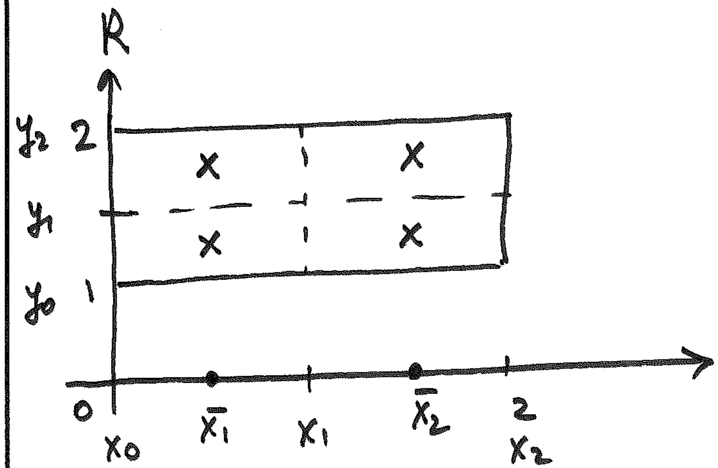
Lecture 29

<u>Ex</u> Use Midpoint Rule w/ $n=m=2$ to approximate

$\iint (x-3y^2)\,dA$ $\qquad R=\{(x,y): 0\le x\le 2,\ 1\le y\le 2\}$



Here

$a=0,\ b=2,\quad c=1,\ d=2$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{2} = 1$

$\Delta y = \frac{d-c}{m} = \frac{2-1}{2} = \frac{1}{2}$

$\bar{x}_i = \frac{x_{i-1}+x_i}{2}, \quad \bar{y}_j = \frac{y_{j-1}+y_j}{2}$

$x_i = x_0 + i\Delta x \qquad y_j = y_0 + j\Delta y$

$x_0 = a = 0 \qquad x_1 = x_0+\Delta x = 0+1 = 1, \quad x_2 = x_0 + 2\Delta x = 2$

$y_0 = c = 1, \quad y_1 = y_0 + \Delta y = 1+\frac{1}{2} = \frac{3}{2}, \quad y_2 = 1+2\cdot\frac{1}{2} = 2 = d$

$\bar{x}_1 = \frac{x_0+x_1}{2} = \frac{1}{2} \qquad \bar{x}_2 = \frac{x_1+x_2}{2} = \frac{3}{2}$

$\bar{y}_1 = \frac{y_0+y_1}{2} = \frac{1+\frac{3}{2}}{2} = \frac{5}{4} \qquad \bar{y}_2 = \frac{y_1+y_2}{2} = \frac{7}{4}$

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$\bar{x}_1 = \frac{1}{2}, \ \bar{x}_2 = \frac{3}{2}, \ \bar{y}_1 = \frac{5}{4}, \ \bar{y}_2 = \frac{7}{4}$

$\Delta A = \Delta x \cdot \Delta y = 1 \cdot \frac{1}{2} = \frac{1}{2}$

$$\iint_R \underbrace{(x-3y^2)}_{f(x,y)} dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A =$$

$$= \left(f(\bar{x}_1, \bar{y}_1) + f(\bar{x}_1, \bar{y}_2) + f(\bar{x}_2, \bar{y}_1) + f(\bar{x}_2, \bar{y}_2) \right) \Delta A =$$

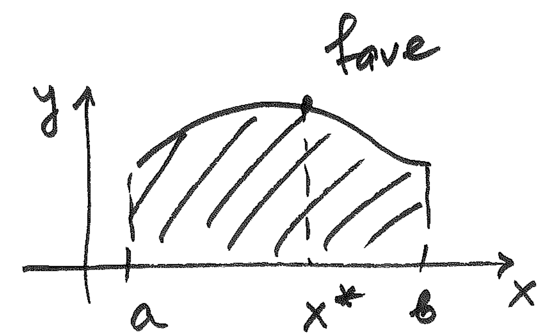
$$= \left(f\left(\tfrac{1}{2}, \tfrac{5}{4}\right) + f\left(\tfrac{1}{2}, \tfrac{7}{4}\right) + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right) + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right) \right) \cdot \frac{1}{2} \approx -11.875$$

Exact value is $\iint_R (x-3y^2)\, dA = -12$

Average Value

Recall for $y = f(x)$, $\quad a \le x \le b$

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx$$



For $z = f(x,y)$

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

here $A(R)$: area of R

Let $f(x,y) \geq 0 \Rightarrow \iint_R f(x,y)\,dA = V$: volume of a solid below surface $z = f(x,y)$ and R

$$\Rightarrow \boxed{V = f_{ave} \cdot A(R)}$$

PROPERTIES OF DOUBLE INTEGRALS

1. $\iint_R [f(x,y) + g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

2. $\iint_R c\,f(x,y)\,dA = c \iint_R f(x,y)\,dA$, c: scalar

3. $f(x,y) \geq g(x,y) \Rightarrow \iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

Iterated Integrals

$f(x,y)$ $\quad R = \{(x,y) : a \leq x \leq b, \quad c \leq y \leq d\}$

Assume x fixed and integrate wrt y

$A(x) = \int_c^d f(x,y)\,dy$: partial integration wrt y

Now $\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$: iterated integral

Notation:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

integration is from inside out

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Ex $\int_0^2 \int_0^{\pi/2} x \sin y \, dy \, dx = \int_{x=0}^{2} \left[\int_{y=0}^{\pi/2} x \sin y \, dy \right] dx =$

$= \int_{x=0}^{2} x \left[\int_{y=0}^{\pi/2} \sin y \, dy \right] dx = \int_{x=0}^{2} x \left[-\cos y \Big|_{y=0}^{\pi/2} \right] dx =$

$= \int_{x=0}^{2} x \left[-\left(\cancel{\cos \tfrac{\pi}{2}} \to 0 - \underset{\overset{\shortparallel}{1}}{\cos 0} \right) \right] dx = \int_0^2 x \, dx = \frac{x^2}{2} \Big|_0^2 = \boxed{2}$

Now

$\int_0^{\pi/2} \int_0^2 x \sin y \, dx \, dy = \int_{y=0}^{\pi/2} \left[\int_{x=0}^{2} x \sin y \, dx \right] dy =$

$= \int_{y=0}^{\pi/2} \sin y \left[\int_{x=0}^{2} x \, dx \right] dy = \int_{y=0}^{\pi/2} \sin y \cdot \underbrace{\left(\frac{x^2}{2} \Big|_0^2 \right)}_{=2} dy = 2 \int_0^{\pi/2} \sin y \, dy =$

$$= 2 \underbrace{(-\cos y)\Big|_0^{\pi/2}}_{=1} = \boxed{2}$$

Note

$$\int_0^2 \int_0^{\pi/2} x \sin y \, dy\, dx = \int_0^{\pi/2} \int_0^2 x \sin y \, dx\, dy$$

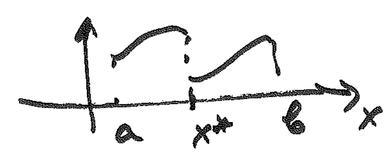
Fubini's Thm

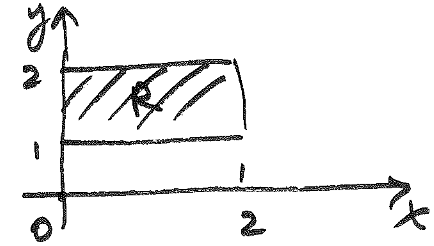
If $f(x,y)$ is continuous on $R = \{(x,y):\ a \le x \le b,\ c \le y \le d\}$

then

$$\boxed{\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy}$$

Note Thm is also valid for piecewise continuous bounded functions.





Ex Evaluate $\iint_R (x-3y^2)\,dA$

where $R = \{(x,y):\ 0 \le x \le 2,\quad 1 \le y \le 2\}$

$$\iint_R (x-3y^2)\,dA = \int_0^2 \int_1^2 (x-3y^2)\,dy\,dx =$$

$$= \int_{x=0}^{2} \left[\int_{y=1}^{2} (x-3y^2)\,dy \right] dx = \int_{x=0}^{2} (xy-y^3)\Big|_{y=1}^{y=2} dx =$$

$$= \int_{x=0}^{2} \left(2x-8-(x-1)\right) dx = \int_0^2 (x-7)\,dx = \left(\frac{x^2}{2} - 7x\right)\Big|_0^2 =$$

$= 2 - 14 = \boxed{-12}$: as was mentioned earlier in the Midpoint Rule example

Note Let $f(x,y) = g(x)\,h(y)$ $\qquad R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx =$$

$$= \int_a^b \Big[\int_c^d \underbrace{g(x)\,h(y)}_{f(x,y)}\,dy \Big]\,dx = \int_a^b g(x) \Big[\underbrace{\int_c^d h(y)\,dy}_{\text{number}} \Big]\,dx =$$

$$= \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

$$\therefore \boxed{\underbrace{\iint_R g(x)\,h(y)\,dA}_{\text{double integral}} = \underbrace{\int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy}_{\text{product of simple integrals}}}$$

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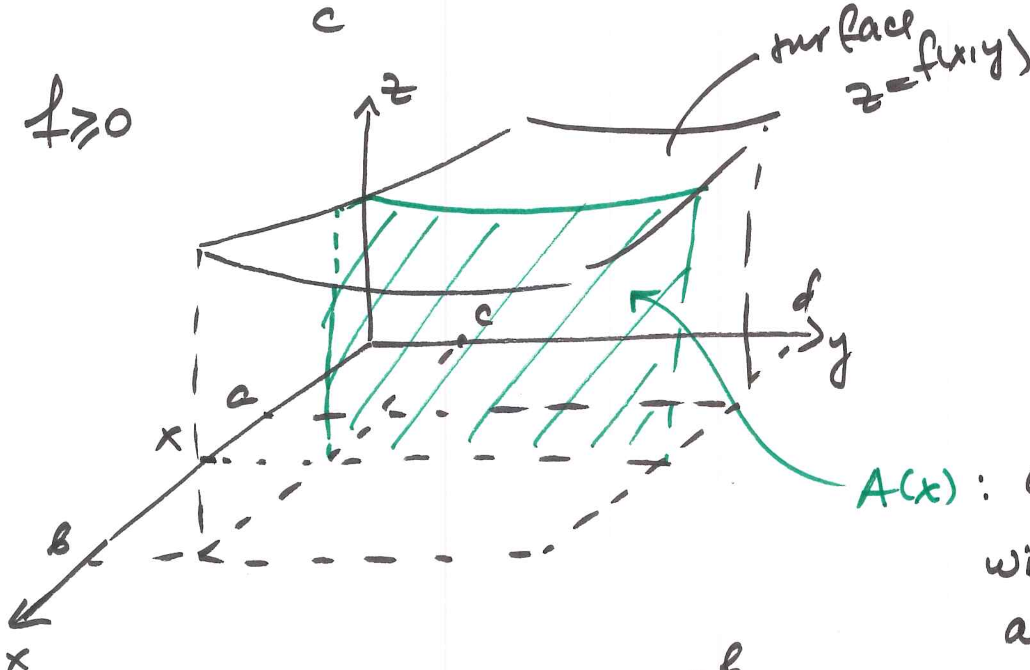
<u>Note</u> $\iint\limits_R f(x,y)\,dA$ $\qquad R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

$A(x) = \int_c^d f(x,y)\,dy$: $\quad x$ is fixed

$f \geq 0$



$\int_a^b A(x)\,dx = \text{Volume}$

$A(x)$: area of cross-section of a solid with plane parallel to yz-plane at some fixed x

Similarly, for $A(y) = \int_a^b f(x,y)\,dx$: $\quad y$ is fixed

$V = \int_c^d A(y)\,dy$