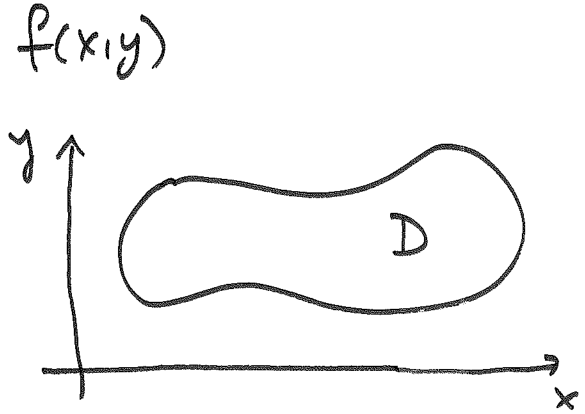
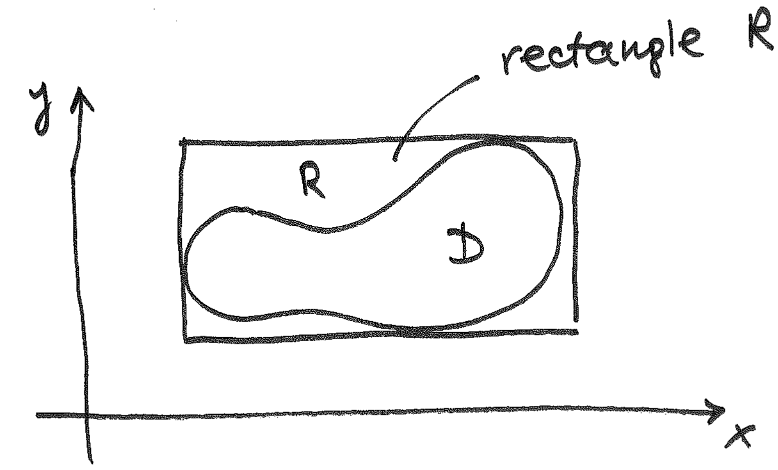
Lecture 30

15.2 Double Integrals over General Domains



$\longrightarrow$



Let

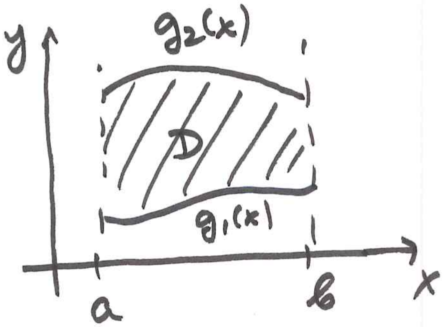
$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \setminus D \end{cases}$$

outside D but inside R

Region of type I:

$D = \{(x,y):\ a \le x \le b,\quad g_1(x) \le y \le g_2(x)\}$

continuous functions



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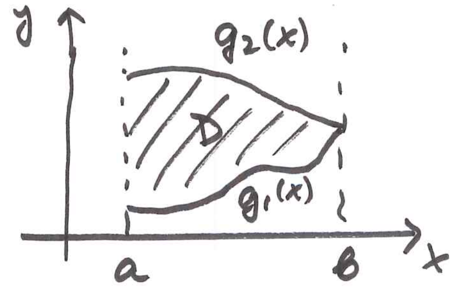

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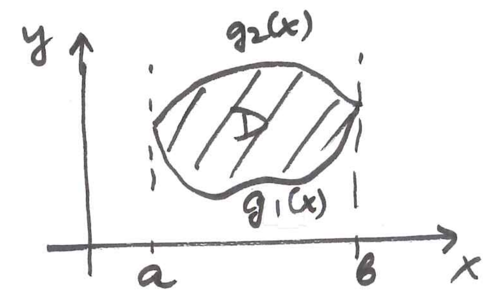
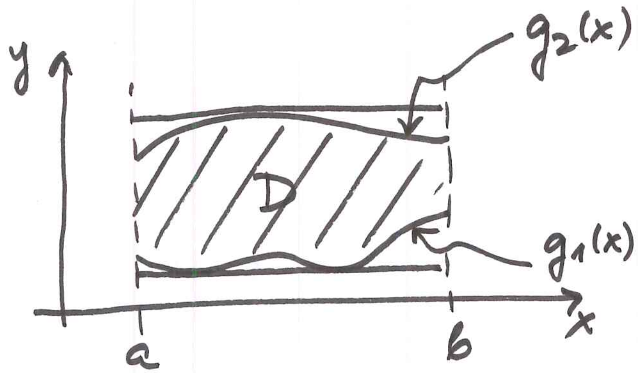
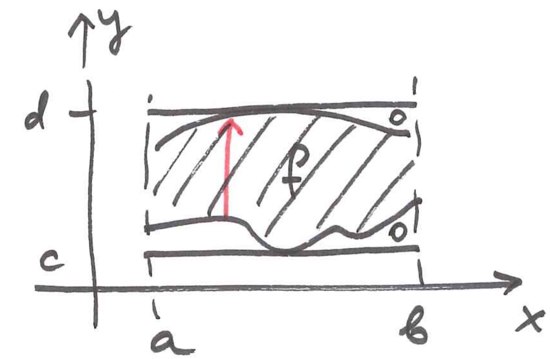


Consider



We define $\iint_D f(x,y)\,dA \overset{\text{def}}{=} \iint_R F(x,y)\,dA$



Then

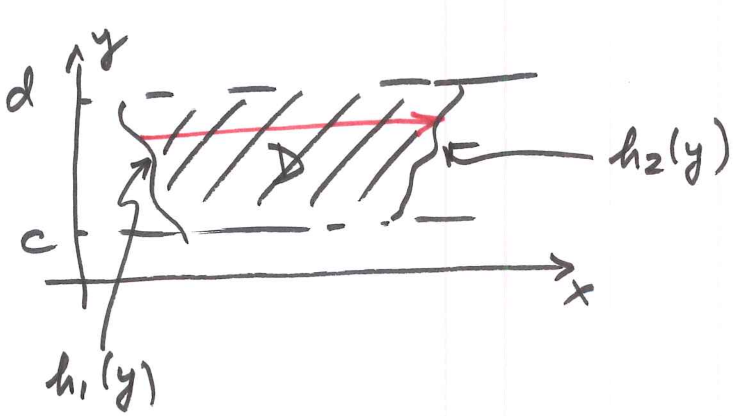
$$\iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$ since $F=0$ for $y<g_1(x)$ and $y>g_2(x)$

$$\therefore \quad \boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

Region of type II :

$D=\{(x,y):\ c\le y\le d,\quad h_1(y)\le x\le h_2(y)$ — continuous functions



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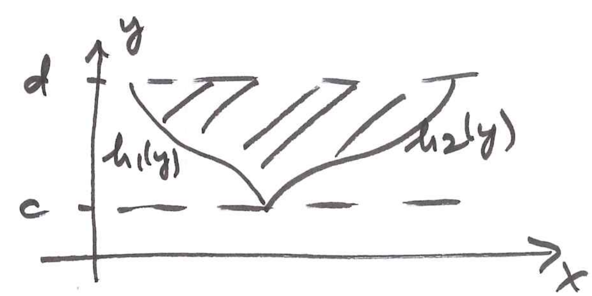
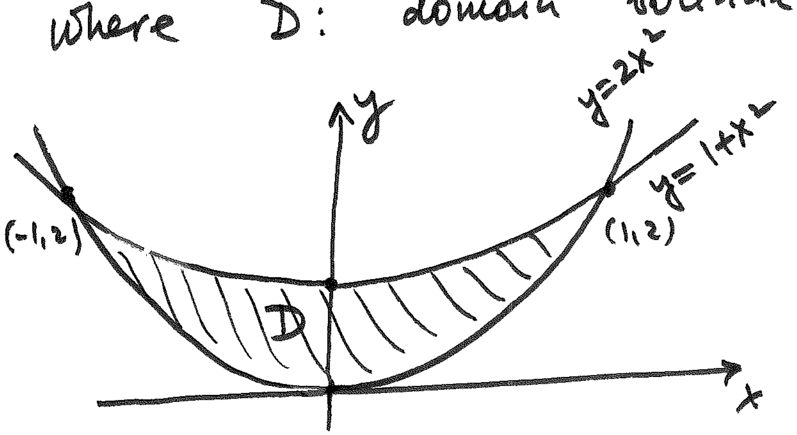


$$\boxed{\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

Ex Compute $\iint_D (x+2y)\,dA$

where D: domain bounded by $y=2x^2$ and $y=1+x^2$



Points of intersection:

$2x^2 = 1+x^2$

$x^2 = 1 \Rightarrow x = \pm 1$

$y = 2(\pm 1)^2 = 2 \Rightarrow (1,2)$ and $(-1,2)$

Region of type I :

$D = \{(x,y) : -1 \le x \le 1, \quad 2x^2 \le y \le 1+x^2\}$

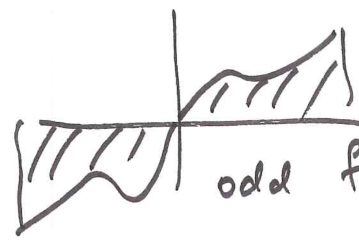
$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{x=-1}^{1} (xy+y^2)\Big|_{y=2x^2}^{y=1+x^2} dx =$$

$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2 - \left(x \cdot 2x^2 + (2x^2)^2\right)\right] dx = \int_{-1}^{1} \left(x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4\right) dx =$$

$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx = \left(-\frac{3}{5}x^5 - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{x^2}{2} + x\right)\Bigg|_{-1}^{1} = \frac{32}{15}$$

no contribution

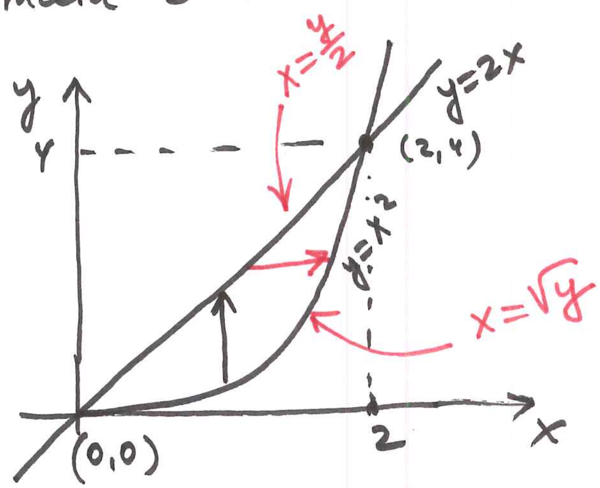
zero contribution



odd function

$\underline{\underline{Ex}}$ Compute $\displaystyle\iint_D (x^2+y^2)\, dA$

Domain D is bounded by $y = 2x$ and $y = x^2$



Intersection: $2x = x^2$

$x(2-x) = 0$

$x_1 = 0, \quad x_2 = 2$

$y_1 = 0, \quad y_2 = 2 \cdot 2 = 4$

$\Rightarrow (0,0)$ & $(2,4)$

Domain of type I: $D = \{(x,y): 0 \le x \le 2, \quad x^2 \le y \le 2x\}$

$$\iint_D (x^2+y^2)\,dA = \int_{x=0}^{2} \int_{y=x^2}^{2x} (x^2+y^2)\,dy\,dx = \dots$$

OR

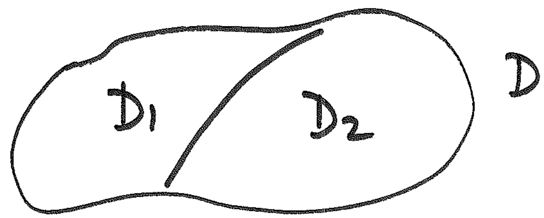
Domain of type II: $D = \{(x,y): 0 \le y \le 4, \quad \frac{y}{2} \le x \le \sqrt{y}\}$

$$\iint_D (x^2+y^2)\,dA = \int_{y=0}^{4} \int_{x=\frac{y}{2}}^{\sqrt{y}} (x^2+y^2)\,dx\,dy = \dots$$

Properties of Double Integrals

1. $$\iint_D \left(f(x,y) + g(x,y)\right) dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

2. $$\iint_D c\,f(x,y)\,dA = c \iint_D f(x,y)\,dA$$

3. 

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

4. $\iint_D 1 \cdot dA = A(D)$: area of domain D

5. If $m \le f(x,y) \le M$ for all $(x,y) \in D$, then

$$m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D)$$

Ex Compute $\iint_D xy\, dA$

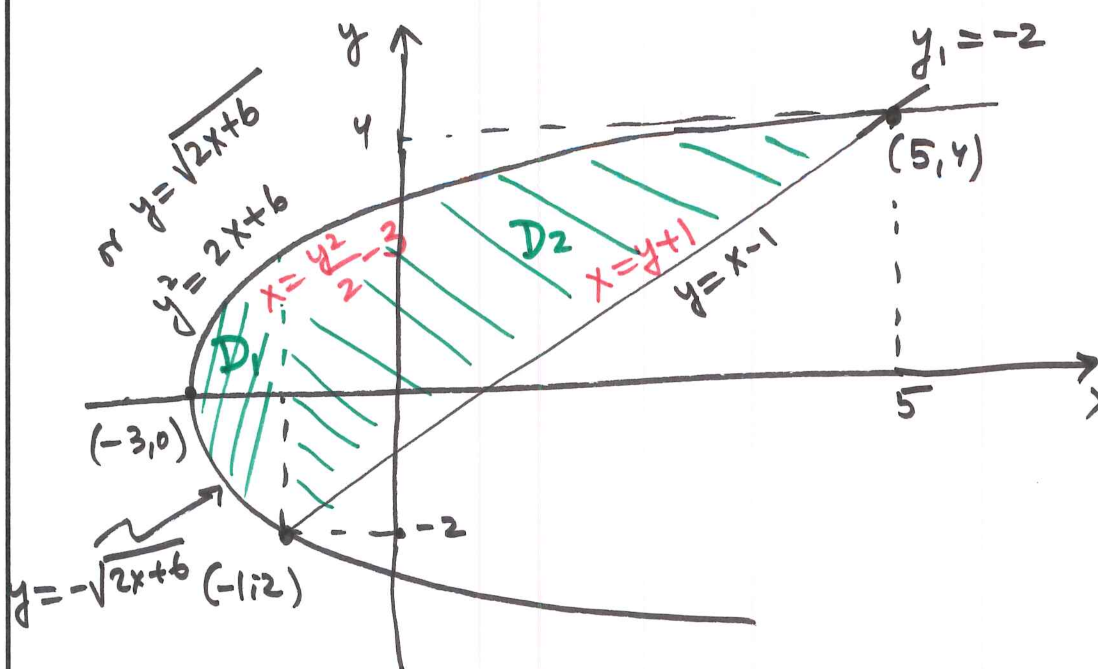
where domain D is bounded by $y = x-1$ and $y^2 = 2x+6$

Intersection:

$(x-1)^2 = 2x+6 \Rightarrow x_1 = -1$ & $x_2 = 5 \Rightarrow (-1,-2)$ & $(5,4)$

$y_1 = -2 \qquad y_2 = 4$

$y^2 = 2x+6$
$2x = y^2 - 6$
$x = \frac{y^2}{2} - 3$



Domain of type II: $D = \{(x,y): -2 \le y \le 4,\ \frac{y^2}{2} - 3 \le x \le y+1\}$

$$\iint_D xy\, dA = \int_{y=-2}^{4} \int_{x=\frac{y^2}{2}-3}^{y+1} xy\, dx\, dy = \ldots$$

Domain of type (I):

$$\iint_D xy\,dA = \underbrace{\int_{x=-3}^{-1}\int_{y=-\sqrt{2x+6}}^{y=\sqrt{2x+6}} xy\,dy\,dx}_{\text{part } D_1} + \underbrace{\int_{x=-1}^{5}\int_{y=x-1}^{\sqrt{2x+6}} xy\,dy\,dx}_{\text{part } D_2}$$