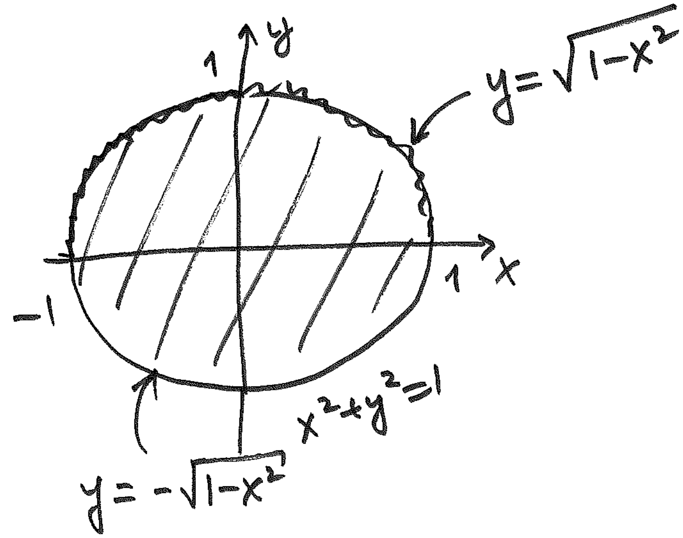
University of Idaho

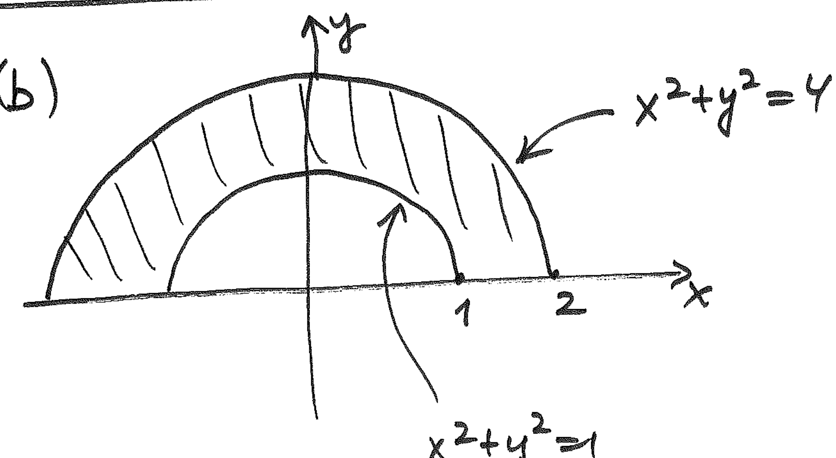
Lecture 31

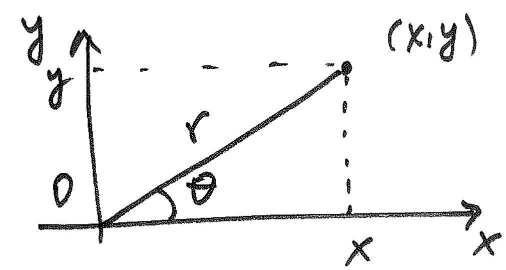
15.3 Double Integrals in Polar Coordinates

(a)



(b)





Polar coordinates: (r, θ)

$x = r\cos\theta, \quad y = r\sin\theta$

$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$

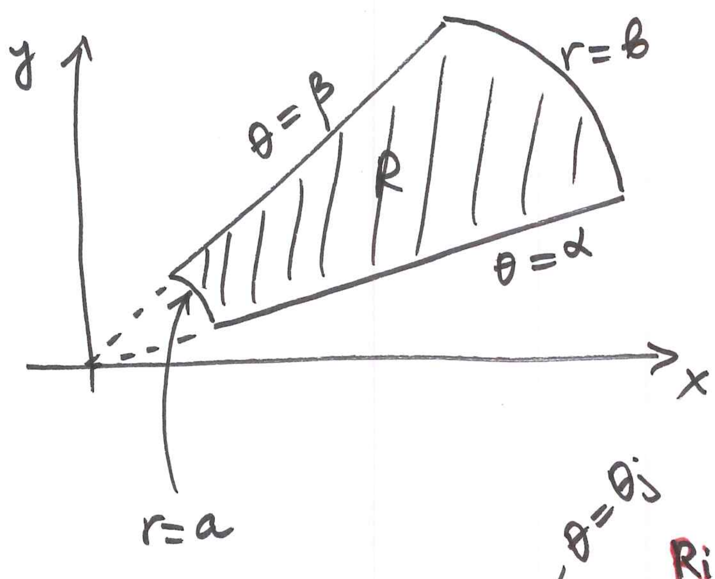
$0 \le \theta \le 2\pi, \quad r \geq 0$

(a) $D = \{(r,\theta): \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}$

(b) $D = \{(r,\theta): \quad 1 \le r \le 2, \quad 0 \le \theta \le \pi\}$

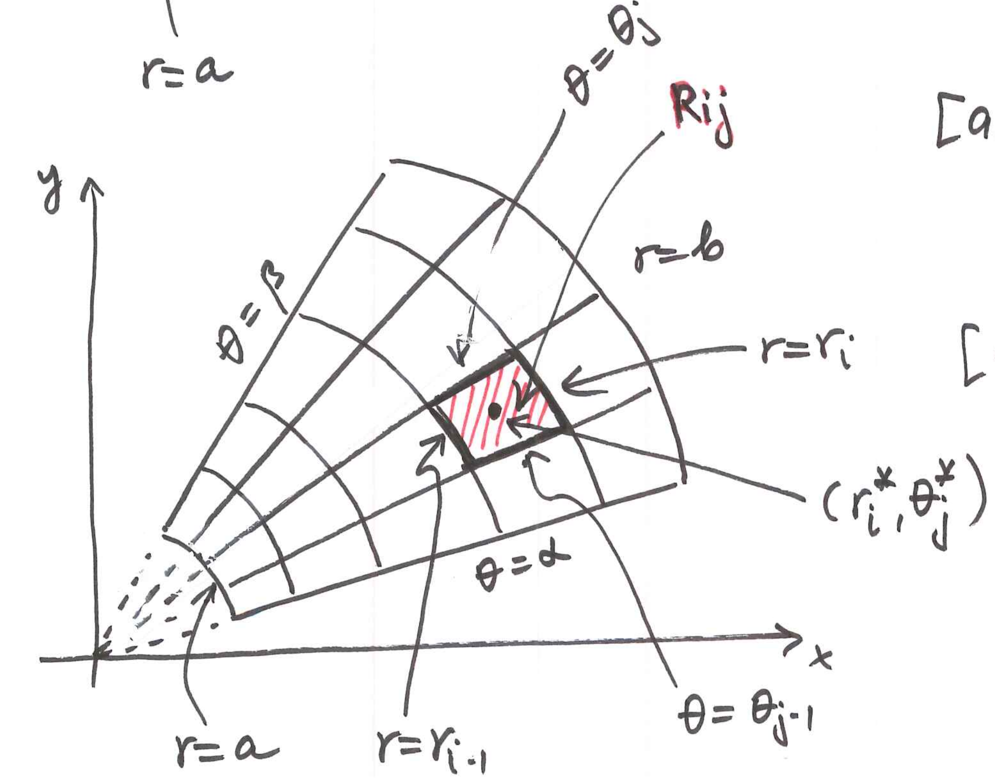
University of Idaho

Polar Rectangle



$$R = \{(r, \theta) : \ a \le r \le b, \quad \alpha \le \theta \le \beta\}$$

$$\iint_R f(x,y)\, dA$$



$[a, b] \to n$ subintervals $[r_{i-1}, r_i]$

$$\Delta r = \frac{b-a}{n}$$

$[\alpha, \beta] \to m$ subintervals $[\theta_{j-1}, \theta_j]$

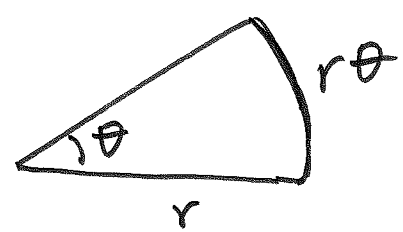
$$\Delta \theta = \frac{\beta - \alpha}{m}$$

University of Idaho

Subrectangle $R_{ij} = \{(r, \theta) : \ r_{i-1} \le r \le r_i, \quad \theta_{j-1} \le \theta \le \theta_j\}$

Sample points: (r_i^*, θ_j^*) : midpoints

$$r_i^* = \frac{r_{i-1} + r_i}{2} \qquad \theta_j^* = \frac{\theta_{j-1} + \theta_j}{2}$$



$r\theta$: arc length

Area of sector : $\frac{1}{2} r \cdot r\theta = \frac{1}{2} r^2 \theta$

$$\Delta A_i = \text{area of } R_{ij} = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta =$$

$$= \underbrace{\frac{1}{2}(r_i + r_{i-1})}_{r_i^*} \underbrace{(r_i - r_{i-1})}_{\Delta r} \Delta\theta = r_i^* \Delta r \Delta\theta$$

University of Idaho

Riemann sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(\underbrace{r_i^* \cos\theta_j^*}_{x}, \underbrace{r_i^* \sin\theta_j^*}_{y}) \underbrace{\Delta A_i}_{r_i^* \Delta r \Delta\theta} =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta \xrightarrow[n,m \to \infty]{}$$

$$\to \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r \, dr \, d\theta$$

$dA = dx\,dy$

Here $dA = r\,dr\,d\theta$

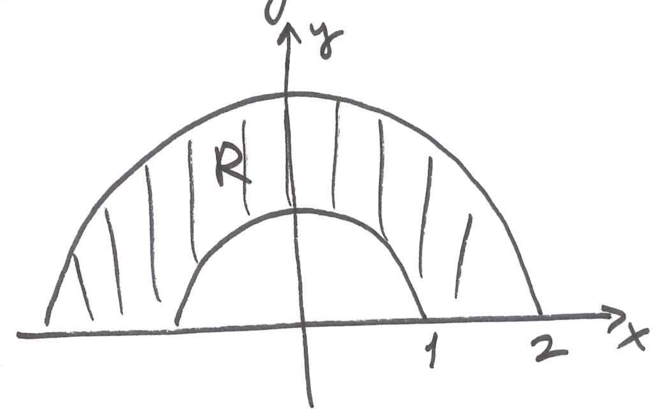
$$\therefore \iint_R f(x,y)\,dA = \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

f is a continuous function on

$R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta,\ 0 \le \beta - \alpha \le 2\pi\}$

Ex Compute $\iint_R (3x + 4y^2)\,dA =$

R: upper half plane bounded by circles $x^2+y^2=1$ and $x^2+y^2=4$



$R = \{(r,\theta) : 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = r\,dr\,d\theta$

University of Idaho

$$= \int_{\theta=0}^{\pi} \int_{r=1}^{2} \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta =$$

$$= \int_{\theta=0}^{\pi} \int_{r=1}^{2} 3r^2\cos\theta\,dr\,d\theta + \int_{\theta=0}^{\pi} \int_{r=1}^{2} 4r^3\sin^2\theta\,dr\,d\theta =$$

$$= \int_{r=1}^{2} 3r^2\,dr \cdot \underbrace{\cancel{\int_{\theta=0}^{\pi} \cos\theta\,d\theta}}_{0} + \int_{r=1}^{2} 4r^3\,dr \cdot \int_{\theta=0}^{\pi} \sin^2\theta\,d\theta \;\boxed{=}$$

$$\int_{r=1}^{2} 3r^2\,dr = r^3\Big|_1^2 \qquad \int_{\theta=0}^{\pi}\cos\theta\,d\theta = \sin\theta\Big|_{\theta=0}^{\pi} \qquad \int_{r=1}^{2} 4r^3\,dr = r^4\Big|_1^2$$

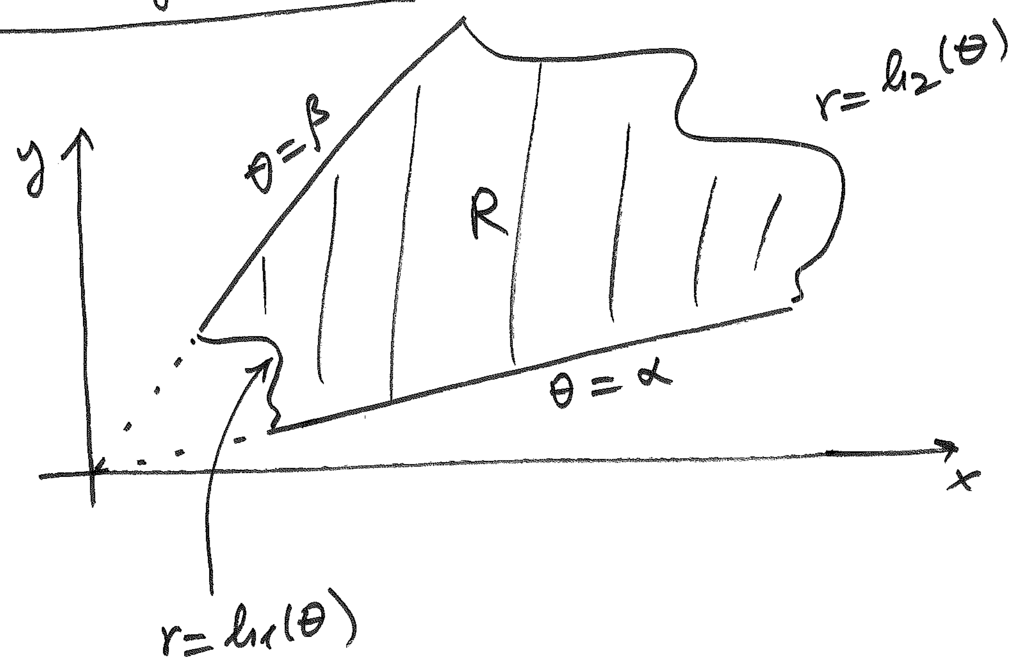
$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

University of Idaho

$$\int_{\theta=0}^{\pi} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi} (1-\cos 2\theta)\, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi} = \frac{\pi}{2}$$

$$\boxed{=} \; r^4\Big|_1^2 \cdot \frac{\pi}{2} = (2^4-1^4)\cdot\frac{\pi}{2} = \boxed{\frac{15\pi}{2}}$$

More generally:

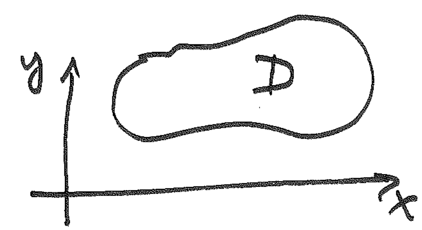


$$\boxed{\iint_R f(x,y)\,dA = \int_{\theta=\alpha}^{\beta} \int_{r=h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta}$$

15.4 Applications of Double Integrals

Ex $\iint_D f(x,y)\,dA$, $f(x,y) \geq 0$

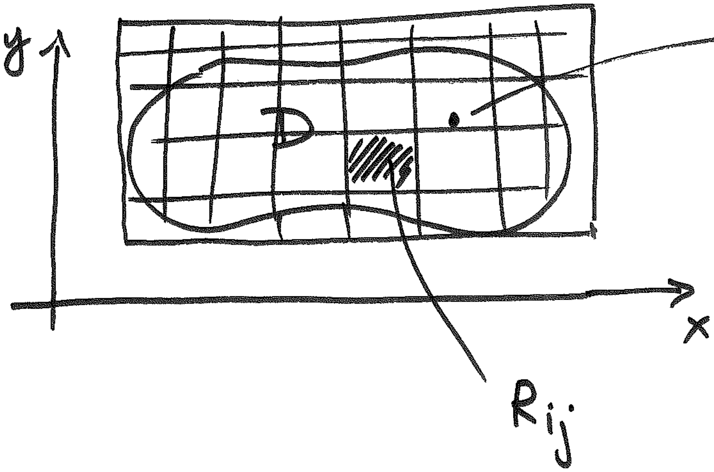
Volume of a solid bounded by surface $z = f(x,y)$ from above and domain D from below.



Density and Mass

Consider lamina : thin plate that occupies region D in $\mathbb{R}^2$, has mass m and density $\rho(x,y)$, rho
$\rho > 0$.

University of Idaho



let $\rho(x,y) = \begin{cases} \rho(x,y) & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$

density in R_{ij} is $\approx$ const

R_{ij} : small subrectangle

(x_{ij}^*, y_{ij}^*) : sample point

mass of R_{ij} is $\approx \rho(x_{ij}^*, y_{ij}^*) \underbrace{\Delta A}_{\text{area of } R_{ij}}$

$\underset{\text{total mass}}{m} \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(x_{ij}^*, y_{ij}^*) \Delta A \xrightarrow[n,m \to \infty]{} \iint_D \rho(x,y)\, dA$

$\therefore \boxed{m = \iint_D \rho(x,y)\, dA}$ <u>total mass of lamina</u>

Ex An electric charge is distributed over region D with charge density $\sigma(x,y)$. Then

$$\boxed{Q = \iint_D \sigma(x,y)\, dA}$$

total charge density

σ : sigma

Moments and Center of Mass

Recall that the moment / 1st moment of a particle of mass m about a given axis is mass × directed distance.

Goal: compute 1st moments of lamina w/ variable density $\rho(x,y)$ wrt coordinate axes.

University of Idaho