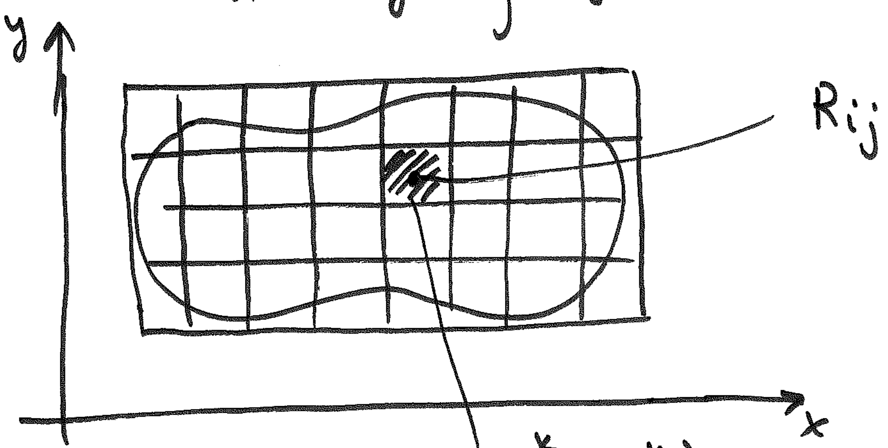
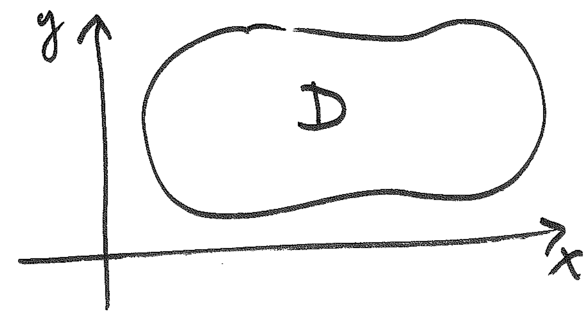
Lecture 32

Moments and Center of Mass

Recall that the moment / 1st moment of a particle of mass m about a given axis = mass x directed distance.

Goal: compute 1st moments of lamina w/ variable density $\rho(x,y)$ wrt coordinate axes.





lamina occupies region D

Moment of R_{ij} wrt x-axis is

$$\approx \Big[\underbrace{\rho(x_{ij}^*, y_{ij}^*)\ \Delta A}_{\text{mass of } R_{ij}}\Big] \cdot \underbrace{y_{ij}^*}_{\substack{\text{distance} \\ \text{to x-axis}}}$$

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Then

$$M_x = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \rho(x_{ij}^*, y_{ij}^*) \cdot y_{ij}^* \, \Delta A =$$

moment of entire lamina wrt x-axis

$$\boxed{M_x = \iint_D \rho(x,y)\, y \, dA}$$

moment of entire lamina wrt x-axis

Similarly,

$$\boxed{M_y = \iint_D \rho(x,y)\, x \, dA}$$

moment of entire lamina wrt y-axis

Coordinates $(\bar{x}, \bar{y})$ of the center of mass of lamina are defined by

$$m\bar{x} = M_y \qquad m\bar{y} = M_x$$

Then

$$\bar{x} = \frac{1}{m} M_y \qquad \bar{y} = \frac{1}{m} M_x$$

where

$$M_x = \iint_D \rho(x,y) \cdot y \, dA \qquad M_y = \iint_D \rho(x,y) \, x \, dA$$

$$m = \iint_D \rho(x,y) \, dA$$

<u>Physically</u>: lamina behaves as if the entire lamina is concentrated at the center of mass

<u>Moments of Inertia</u> (<u>2nd moments</u>)

Moment of inertia of a particle of mass m about a given axis is mr^2, where r is the distance from the particle to the axis.

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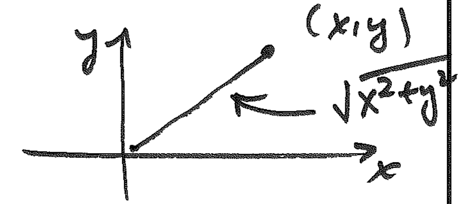
$$I_x = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\rho(x_{ij}^*, y_{ij}^*)\, \Delta A\right] \cdot (y_{ij}^*)^2$$

$$\boxed{I_x = \iint_D y^2 \rho(x,y)\, dA}$$

moment of inertia of lamina wrt x-axis

$$\boxed{I_y = \iint_D x^2 \rho(x,y)\, dA}$$

moment of inertia of lamina wrt y-axis



$$I_0 = I_x + I_y = \iint_D (x^2+y^2)\rho(x,y)\, dA$$

moment of inertia wrt origin

or polar moment of inertia

Radius R of gyration is the number:

$$mR^2 = I$$

where

m: total mass

I: moment of inertia about an axis

about x-axis: $m\bar{\bar{y}}^2 = I_x$

about y-axis: $m\bar{\bar{x}}^2 = I_y$

The pt $(\bar{\bar{x}}, \bar{\bar{y}})$: at which mass of lamina can be concentrated w/o changing moments of inertia wrt coordinate axes.

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Expected Values

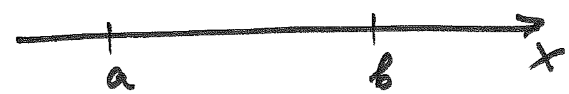
X : random variable

$f(x)$: its probability density function

$f(x) \geq 0 \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1$

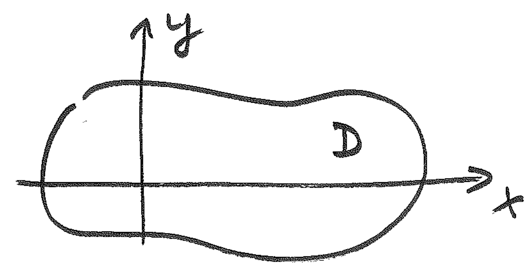
$P\{a \leq X \leq b\} = \int_a^b f(x)\,dx$: probability that a random variable X assumes values between a and b

$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$: mean of X



Now consider X, Y: two random variables.

$f(x,y)$: joint probability density function



$$P\{(X,Y) \in D\} = \iint_D f(x,y)\,dA$$

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$$\left.\begin{array}{l}\mu_1 = \iint\limits_{\mathbb{R}^2} x\, f(x,y)\, dA \;:\; X\text{-mean} \\ \mu_2 = \iint\limits_{\mathbb{R}^2} y\, f(x,y)\, dA \;:\; Y\text{-mean}\end{array}\right\} \text{expected values of } X \text{ and } Y$$

Note $\iint\limits_{\mathbb{R}^2} f(x,y)\, dA = 1$ $f(x,y) \geq 0$

Recall

$$M_x = \iint\limits_{D} y\, \rho(x,y)\, dA \;:\; \text{moment about } x\text{-axis}$$

$$M_y = \iint\limits_{D} x\, \rho(x,y)\, dA \;:\; \text{moment about } y\text{-axis}$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint\limits_{D} x\, \rho(x,y)\, dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint\limits_{D} y\, \rho(x,y)\, dA$$

$$m = \iint\limits_{D} \rho(x,y)\, dA$$

<u>Note</u> μ_1, μ_2 resemble moments of lamina w/ density $\rho(x,y)$.

We can interprete probability as continuously distributed mass

$$\Rightarrow \iint_{\mathbb{R}^2} f(x,y)\, dA \; : \quad \text{total "probability mass"}$$

$\bar{x}, \bar{y}$ show that we can think of expected values μ_1 and μ_2 as the "center of mass" of probability distribution.

In some cases we can write

$$f(x,y) = f_1(x) \cdot f_2(y) \Rightarrow X \text{ and } Y \text{ are independent variables}$$

$f_1(x)$: probability density function of X

$f_2(y)$: probability density function of Y

$$\iint_D f(x,y)\,dA = \int f_1(x)\,dx \cdot \int f_2(y)\,dy$$

double $\int$ product of two single integrals

Special case: $D = R = \{a \le x \le b,\ c \le y \le d\}$

Then

$$P\{a \le X \le b,\ c \le Y \le d\} = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

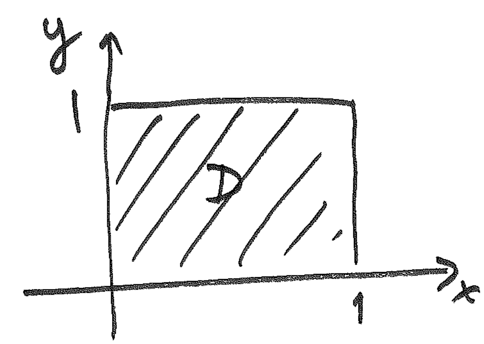
if X, Y are independent $\Rightarrow$

$$P\{a \le X \le b,\ c \le Y \le d\} = \int_a^b f_1(x)\,dx \cdot \int_c^d f_2(y)\,dy$$

Ex Verify that

$$f(x,y) = \begin{cases} 4xy, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

is a joint density function.



Solution

$f(x,y) \geq 0$? yes for $(x,y) \in \mathbb{R}^2$

$$\iint_{\mathbb{R}^2} f(x,y) \stackrel{?}{=} 1$$

$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_{x=0}^{1} \int_{y=0}^{1} 4xy\, dy\, dx =$$

$$= 4 \int_0^1 x\, dx \cdot \int_0^1 y\, dy = 4 \left.\frac{x^2}{2}\right|_0^1 \cdot \left.\frac{y^2}{2}\right|_0^1 = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1 \quad ✓$$

$\therefore$ $f(x,y)$ is a joint density function

(b) if X and Y are random variables whose joint density function as in a), find

(i) $P(X \geq \frac{1}{2})$

(ii) $P(X \geq \frac{1}{2},\ Y \leq \frac{1}{2})$

(iii) $P(X + Y \geq 1)$

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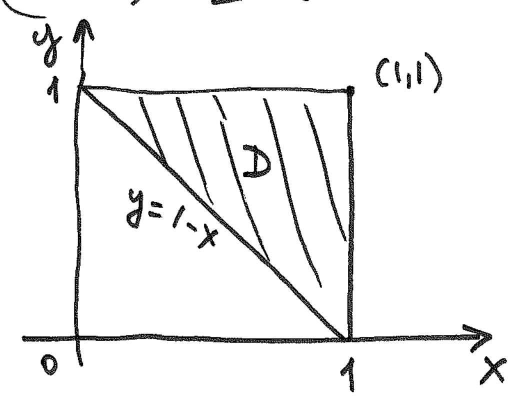
(i) $P(X \geq \frac{1}{2}) = \int_{x=\frac{1}{2}}^{\infty} \int_{y=-\infty}^{\infty} f(x,y)\, dy\, dx = \int_{\frac{1}{2}}^{1} \int_{0}^{1} 4xy\, dy\, dx =$

$= 4 \int_{\frac{1}{2}}^{1} x\, dx \cdot \int_{0}^{1} y\, dy = 4 \frac{x^2}{2} \Big|_{\frac{1}{2}}^{1} \cdot \frac{y^2}{2} \Big|_{0}^{1} = 2\left(1 - \frac{1}{4}\right) \cdot \frac{1}{2} = \boxed{\frac{3}{4}}$

(ii) $P(X \geq \frac{1}{2}, Y \leq \frac{1}{2}) = \int_{x=\frac{1}{2}}^{\infty} \int_{y=-\infty}^{\infty} f(x,y)\, dy\, dx = \int_{x=\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} 4xy\, dy\, dx =$

$= 4 \int_{\frac{1}{2}}^{1} x\, dx \cdot \int_{0}^{\frac{1}{2}} y\, dy = 4 \frac{x^2}{2} \Big|_{\frac{1}{2}}^{1} \cdot \frac{y^2}{2} \Big|_{0}^{\frac{1}{2}} = 2\left(1 - \frac{1}{4}\right) \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{4} \cdot \frac{1}{4} = \boxed{\frac{3}{16}}$

(iii) $P(X + Y \geq 1) = \int_{x=0}^{1} \int_{y=1-x}^{1} 4xy\, dy\, dx =$ double integral w/ region of type I



$x + y \geq 1$ or $y \geq 1 - x$

or $\int_{y=0}^{1} \int_{x=1-y}^{1} 4xy\, dx\, dy$ type II region

$$= 4\int_{x=0}^{1} x \int_{y=1-x}^{1} y\,dy\,dx = 4\int_{x=0}^{1} x \cdot \frac{y^2}{2}\Big|_{y=1-x}^{1} dx = 2\int_0^1 x\big(1-\underbrace{(1-x)^2}_{1-2x+x^2}\big)\,dx =$$

$$= 2\int_0^1 x(2x-x^2)\,dx = 2\int_0^1 (2x^2-x^3)\,dx = 2\left(x^3-\frac{x^4}{4}\right)\Big|_0^1 = 2\left(1-\frac{1}{4}\right) = \boxed{\frac{3}{2}}$$

Ex Waiting time is modeled by

$$f(t) = \begin{cases} 0, & t<0 \\ \mu^{-1} e^{-t/\mu}, & t \geq 0 \end{cases}$$ exponential density function

μ: mean waiting time