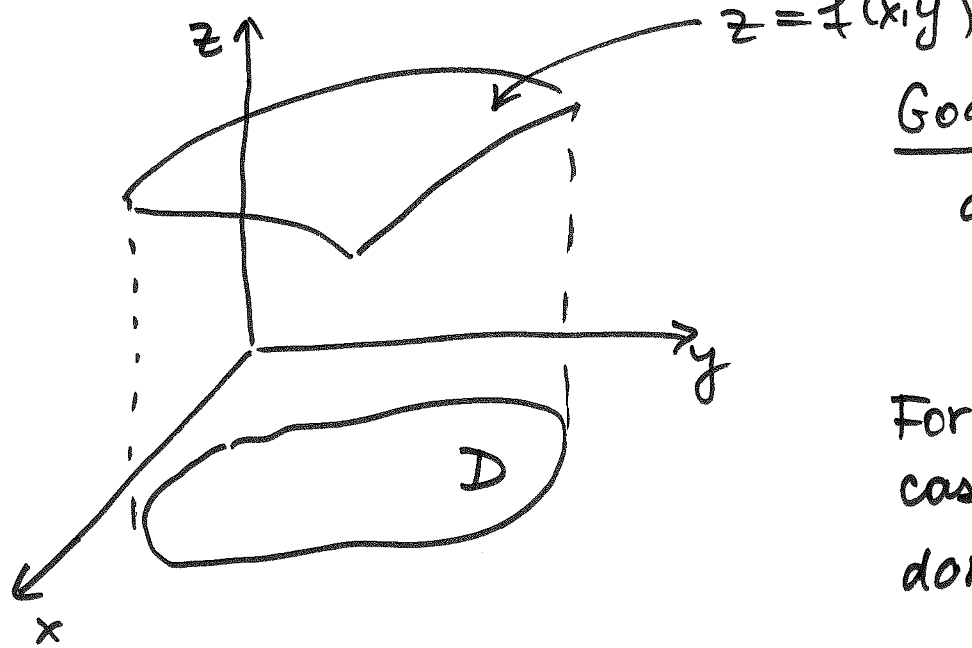
Lecture 33

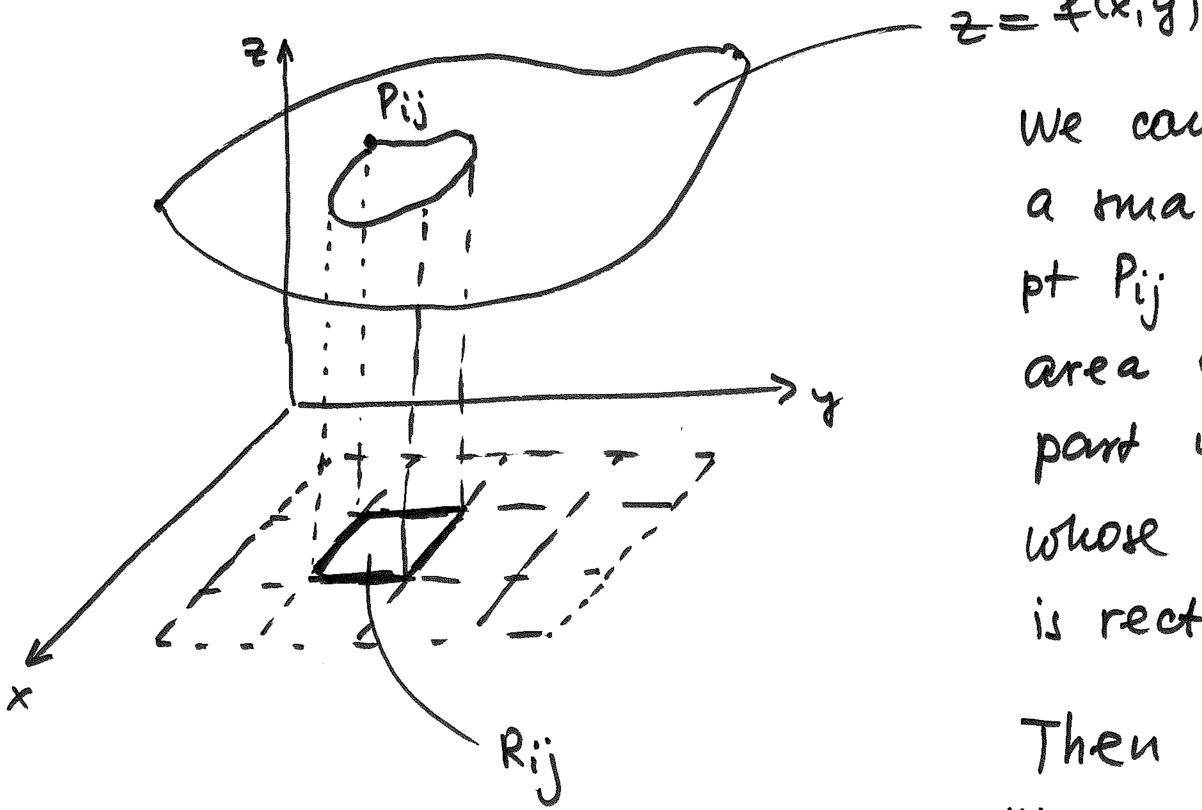
15.5 Surface Area

Consider surface $z = f(x,y)$.



Goal: compute surface area above some region D, i.e. $(x,y) \in D$.

For simplicity, we consider the case when D is a rectangular domain R.



We can approximate area of a small piece w/ corner at pt P_{ij} by ΔT_{ij} that is area of parallelogram, a part of tangent plane at P_{ij}, whose projection on xy plane is rectangle R_{ij}.

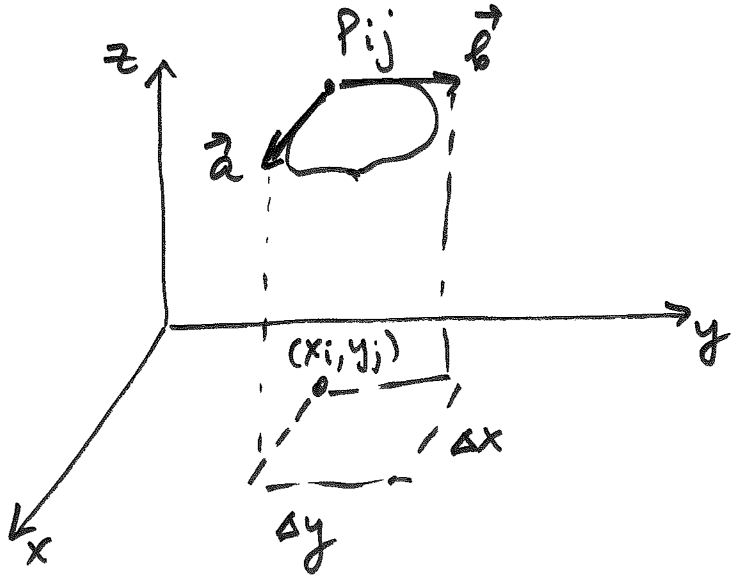
Then

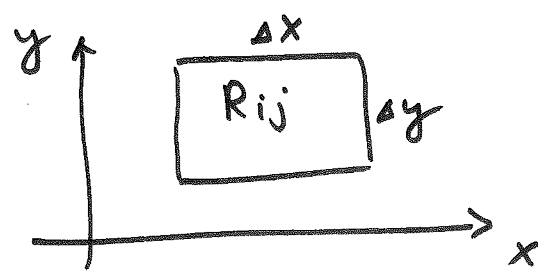
total surface area $\quad A(S) \approx \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta T_{ij}$

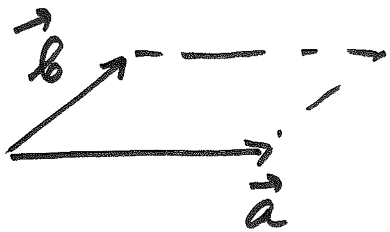
and

$$A(S) = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta T_{ij} \quad : \quad \text{total surface area}$$

University of Idaho





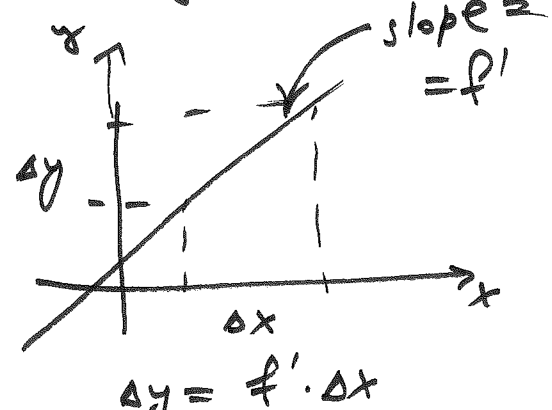


Area of parallelogram determined by $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$

Recall, $f_x(x_i, y_j)$, $f_y(x_i, y_j)$ are slopes of tangent lines through P_{ij} in the direction of $\vec{a}$, $\vec{b}$.

$$\vec{a} = \Delta x \, \hat{i} + f_x(x_i, y_j)\,\Delta x \, \hat{k}$$

$$\vec{b} = \Delta y \, \hat{j} + f_y(x_i, y_j)\,\Delta y \, \hat{k}$$



Y

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix} =$$

$$= \hat{i}\left(-f_x(x_i, y_j)\,\Delta x \Delta y\right) - \hat{j} \cdot f_y(x_i, y_j)\,\Delta x \Delta y + \hat{k}\,\Delta x \Delta y$$

$$\therefore \quad \Delta T_{ij} = |\vec{a} \times \vec{b}| = \sqrt{\left(f_x(x_i, y_j)\right)^2 + \left(f_y(x_i, y_j)\right)^2 + 1} \cdot \Delta x \Delta y$$

$$\therefore \quad A(S) = \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sqrt{\left(f_x(x_i, y_j)\right)^2 + \left(f_y(x_i, y_j)\right)^2 + 1} \cdot \Delta x \Delta y$$

i.e. $$\boxed{A(S) = \iint_R \sqrt{\left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2 + 1} \; dx\,dy}$$

total surface area of $z = f(x,y)$ above R or D

R or D: arbitrary domain

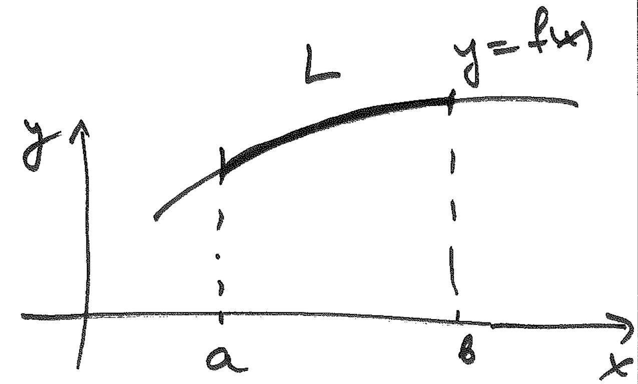
University of Idaho

Recall $z = f(x,y)$. Then we can write

$$A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy$$

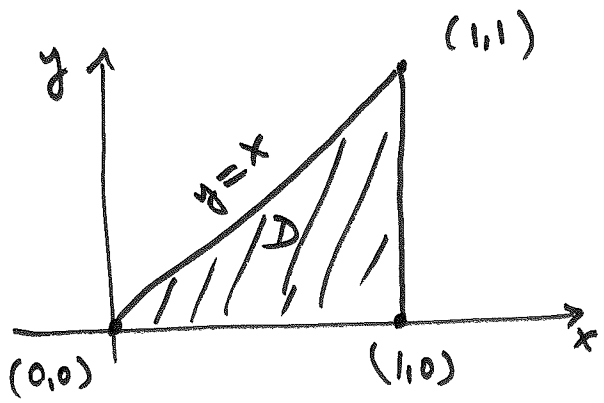
Compare this result with:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



<u>Ex</u> Find surface area of the part of surface $z = x^2 + 2y$ that lies above triangular region w/ vertices $(0,0)$, $(1,0)$ and $(1,1)$

<u>Solution</u>



$f(x,y) = x^2 + 2y$

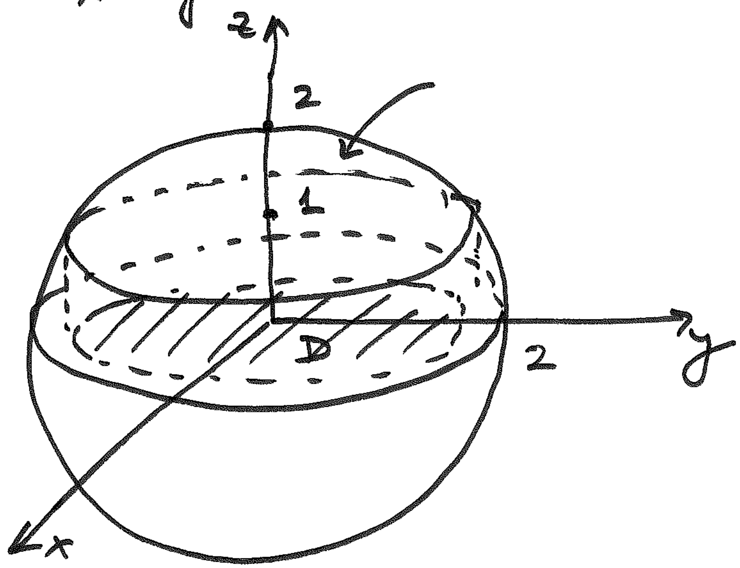
$f_x = 2x \qquad f_y = 2$

$$A(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA =$$

$$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} \sqrt{(2x)^2 + 2^2 + 1}\, dy\, dx = \dots$$

Ex Find the surface area of the part of sphere $x^2+y^2+z^2=4$ that lies above the plane $z=1$.



Intersection of sphere w/ plane $z=1$.

Set $z=1$ in $x^2+y^2+z^2=4$

$$x^2+y^2+1=4$$

or $x^2+y^2=3$: circle of rad. 3 centered at (0,0)

$\therefore\ D = \{(x,y) : x^2+y^2 \le 3\}$

$x^2+y^2+z^2=4$

$z^2=4-x^2-y^2$

$z=+\sqrt{4-x^2-y^2}$: top hemisphere

$\Rightarrow \quad f=\sqrt{4-x^2-y^2}$

$f_x = \dfrac{-2x}{2\sqrt{4-x^2-y^2}} = \dfrac{-x}{\sqrt{4-x^2-y^2}} \qquad f_y = \dfrac{-y}{\sqrt{4-x^2-y^2}}$

$$\therefore A(S) = \iint\limits_{\substack{D \\ x^2+y^2\le 3}} \sqrt{\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2 + 1}\; dA =$$

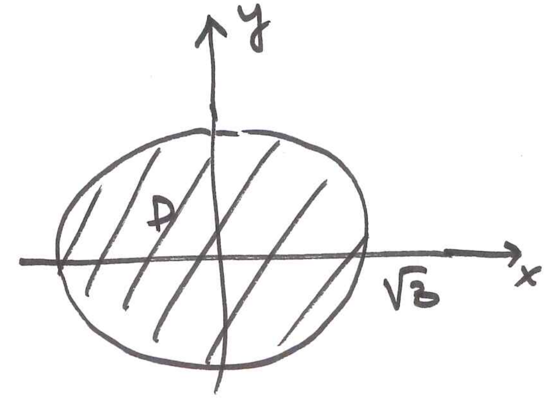
$$= \iint\limits_{x^2+y^2\le 3} \sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1}\; dA =$$

University of Idaho

$$= \iint\limits_{x^2+y^2\le 3} \sqrt{\frac{\cancel{x^2}+\cancel{y^2}+4-\cancel{x^2}-\cancel{y^2}}{4-x^2-y^2}}\, dA = \iint\limits_{x^2+y^2\le 3} \frac{2}{\sqrt{4-x^2-y^2}}\, dA \overset{\text{polar}}{\underset{\text{coordinates}}{=}}$$

$x = r\cos\theta, \quad y = r\sin\theta$

$x^2+y^2 = r^2$



$D = \{(r,\theta): \ 0 \le r \le \sqrt{3}, \quad 0 \le \theta \le 2\pi\}$

$dA = r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}\, r\,dr\,d\theta = \left| \begin{array}{l} u = 4-r^2 \\ du = -2r\,dr \\ r=0 \Rightarrow u=4 \\ r=\sqrt{3} \Rightarrow u=1 \end{array} \right| =$$

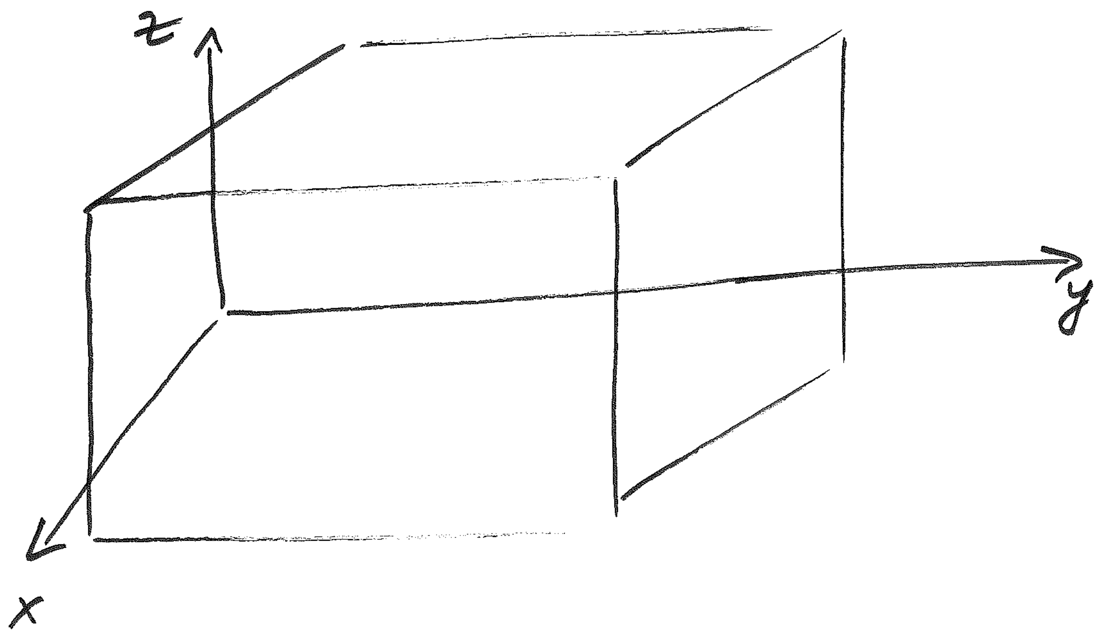
$$= \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{2\pi} \cdot \int_{u=4}^{1} \frac{-du}{\sqrt{u}} = 2\pi \cdot \int_{1}^{4} \frac{du}{\sqrt{u}} \;\boxed{=}$$

Note $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$= 2\pi \cdot 2\sqrt{u}\Big|_1^4 = 4\pi(2-1) = \boxed{4\pi}$

15.6 Triple Integrals

$B = \{(x, y, z) : \ a \le x \le b, \ c \le y \le d, \ r \le z \le s\}$



Divide B into sub-boxes.

$[a,b] \rightarrow \ell$ subintervals $[x_{i-1}, x_i]$, $\Delta x = \frac{b-a}{\ell}$

$[c,d] \rightarrow n$ subintervals $[y_{j-1}, y_j]$, $\Delta y = \frac{d-c}{n}$

$[r,s] \rightarrow m$ subintervals $[z_{k-1}, z_k]$, $\Delta z = \frac{s-r}{m}$