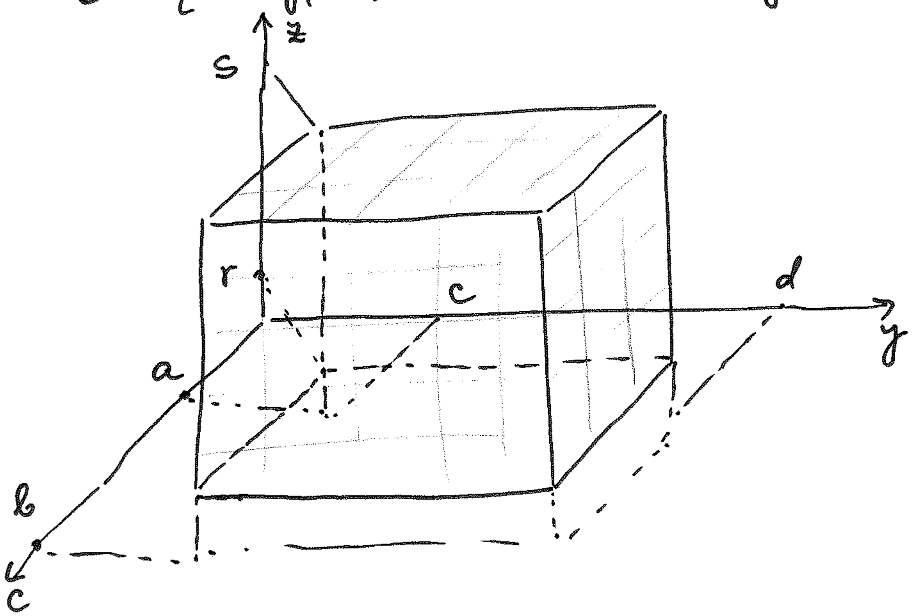
Lecture 34

15.6 Triple Integrals

$B = \{(x, y, z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$



Divide B into sub-boxes.

$[a, b] \to l$ subintervals $[x_{i-1}, x_i]$ of length $\Delta x = \frac{b-a}{l}$

$[c, d] \to n$ subintervals $[y_{j-1}, y_j]$ of length $\Delta y = \frac{d-c}{n}$

$[r, s] \to m$ subintervals $[z_{k-1}, z_k]$ of length $\Delta z = \frac{s-r}{m}$

Sub-box

$$B_{ijk} = \{(x, y, z) : x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j,\ z_{k-1} \le z \le z_k\}$$

ΔV = volume of B_{ijk} is $\Delta x \Delta y \Delta z$

$(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$: sample pt in B_{ijk}

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$$\sum_{i=1}^{l}\sum_{j=1}^{n}\sum_{k=1}^{m} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$: triple Riemann sum

We define triple integral of $f(x,y,z)$ over rectangular box B:

$$\iiint_B f(x,y,z)\,dV \overset{\text{def}}{=} \lim_{l,n,m\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{n}\sum_{k=1}^{m} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

if limit exists.

<u>Note</u> This $\iiint$ exists if $f(x,y,z)$ is continuous.

<u>Fubini's Thm for triple integrals</u>

If $f(x,y,z)$ is continuous on a rectangular box $B=[a,b]\times[c,d]\times[r,s]$, then

$$\underbrace{\iiint_B f(x,y,z)\,dV}_{\text{triple }\int} = \underbrace{\int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz}_{\text{iterated }\int} =$$

6 possible directions of integration

$$= \int_a^b\int_c^d\int_r^s f(x,y,z)\,dz\,dy\,dx = \int_a^b\int_r^s\int_c^d f(x,y,z)\,dy\,dz\,dx$$

Triple Integrals over a general bounded domain E

Enclose domain E into a rectangular box B and define

$$F(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \in E \\ 0 & (x,y,z) \in B \setminus E \quad \text{(otherwise)} \end{cases}$$

$$\therefore \quad \iiint_E f(x,y,z)\, dV = \iiint_B F(x,y,z)\, dV$$

<u>Note</u>: this $\iiint$ exists if $f(x,y,z)$ is continuous in E and boundary of E is piecewise smooth.

Solid region E is of <u>type 1</u>: solid E lies between two continuous surfaces $u_1(x,y)$ and $u_2(x,y)$:

$$E = \{ (x,y,z) : \underbrace{(x,y) \in D}_{\text{projection of } E \text{ onto } xy\text{-plane}}, \quad u_1(x,y) \le z \le u_2(x,y) \}$$

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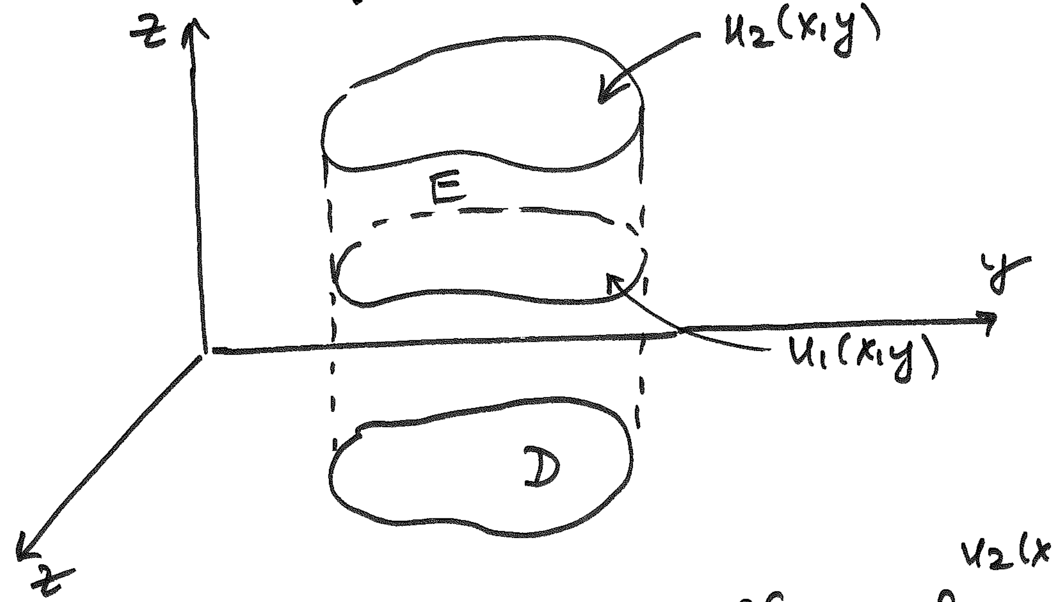
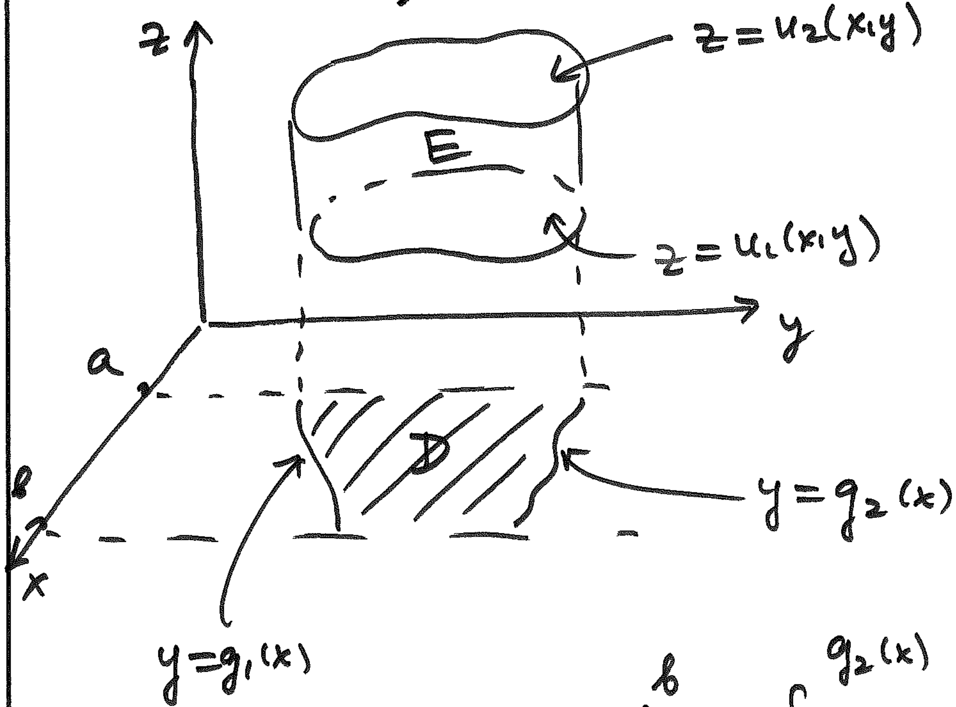


$$\therefore \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{z=u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

<u>Particular case</u> : D is of type I region :

$$D = \{(x,y) : a \le x \le b,\ \ g_1(x) \le y \le g_2(x)\}$$

Solid $E = \{(x,y,z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ \ u_1(x,y) \le z \le u_2(x,y)\}$



$dV = dx\,dy\,dz$

$$\iiint_E f(x,y,z)\,dV = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Particular case: D is a domain of type $\overline{\underline{II}}$:

$D = \{(x,y):\ h_1(y) \le x \le h_2(y),\quad c \le y \le d\}$

$E = \{(x,y,z):\ h_1(y) \le x \le h_2(y),\ c \le y \le d,\quad u_1(x,y) \le z \le u_2(x,y)\}$

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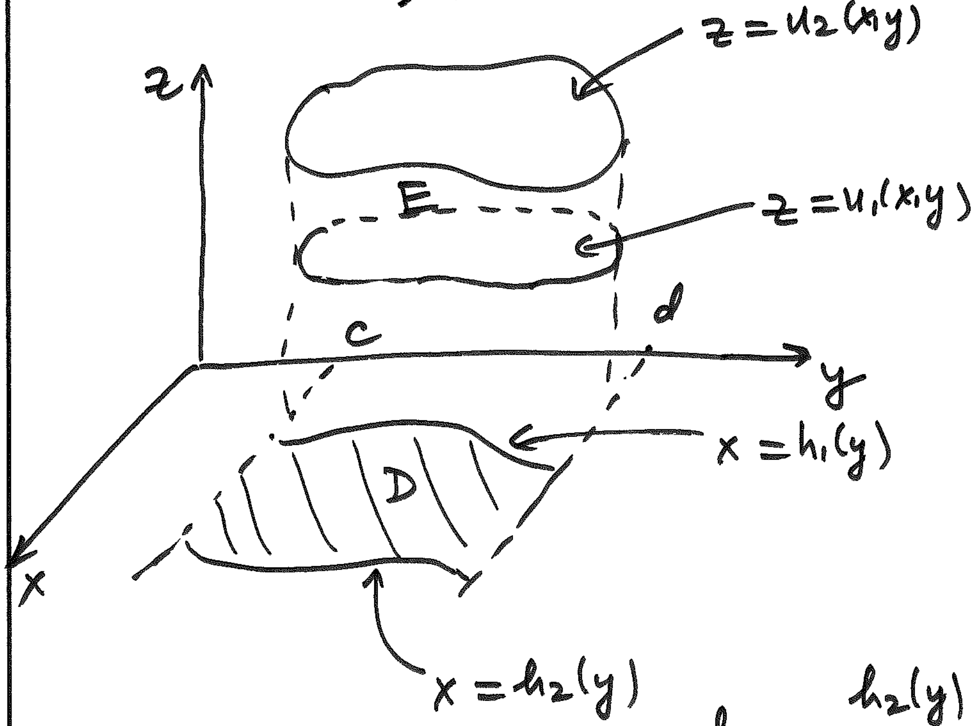

$$\iiint_E f(x,y,z)\,dV = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} \int_{z=u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Ex Evaluate $\iiint_E z\,dV$

E: solid tetrahedron bounded by four planes: $\underbrace{x=0}_{yz\text{-plane}}$, $\underbrace{y=0}_{xz\text{-plane}}$, $\underbrace{z=0}_{xy\text{-plane}}$ and $x+y+z=1$

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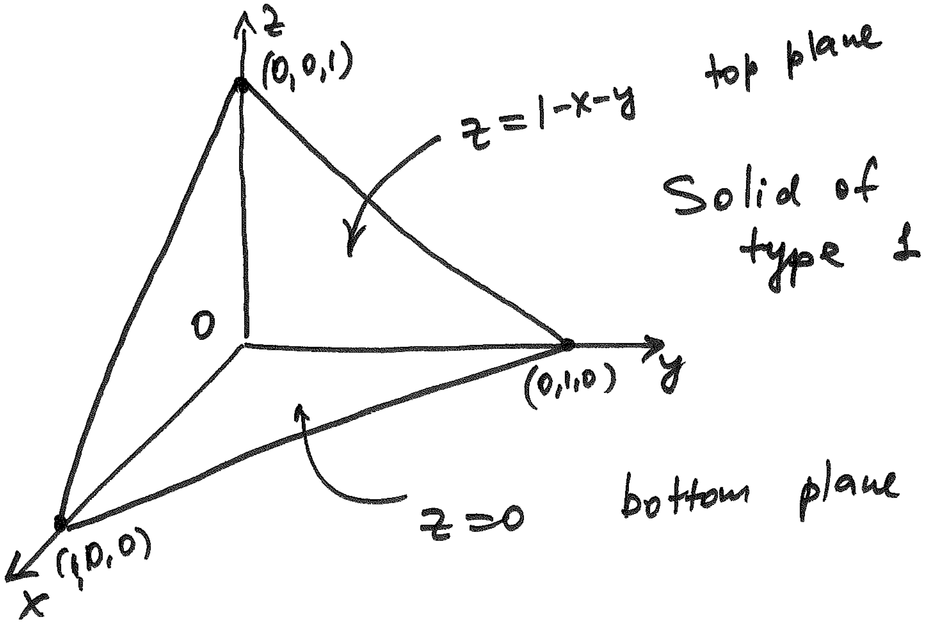
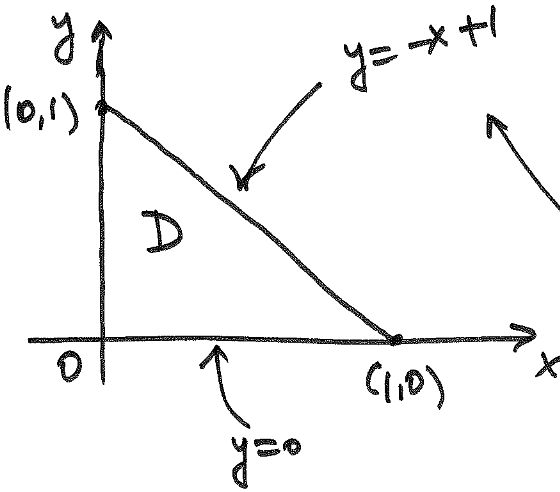


x-intercept: $y=z=0 \Rightarrow x=1 \Rightarrow (1,0,0)$

y-intercept: $x=z=0 \Rightarrow y=1 \Rightarrow (0,1,0)$

z-intercept: $x=y=0 \Rightarrow z=1 \Rightarrow (0,0,1)$



D can be regarded as domain of type I, for example:

$$D=\{(x,y):\ 0\le x\le 1,\ \ 0\le y\le -x+1\}$$

OR: plane $x+y+z=1$

projection onto xy-plane: $z=0$

$\Rightarrow x+y=1 \Rightarrow y=1-x$: the same

Hence,

$$E = \{(x,y,z):\ 0 \le x \le 1,\ 0 \le y \le -x+1,\quad 0 \le z \le 1-x-y\}$$

$$\iiint_E z\,dV = \int_{x=0}^{1} \int_{y=0}^{-x+1} \int_{z=0}^{1-x-y} z\,dz\,dy\,dx =$$

$$= \int_{x=0}^{1} \int_{y=0}^{-x+1} \frac{1}{2} z^2 \Big|_{z=0}^{z=1-x-y} dy\,dx = \int_{x=0}^{1} \int_{y=0}^{-x+1} \frac{1}{2}\left[(1-x-y)^2 - 0^2\right] dy\,dx =$$

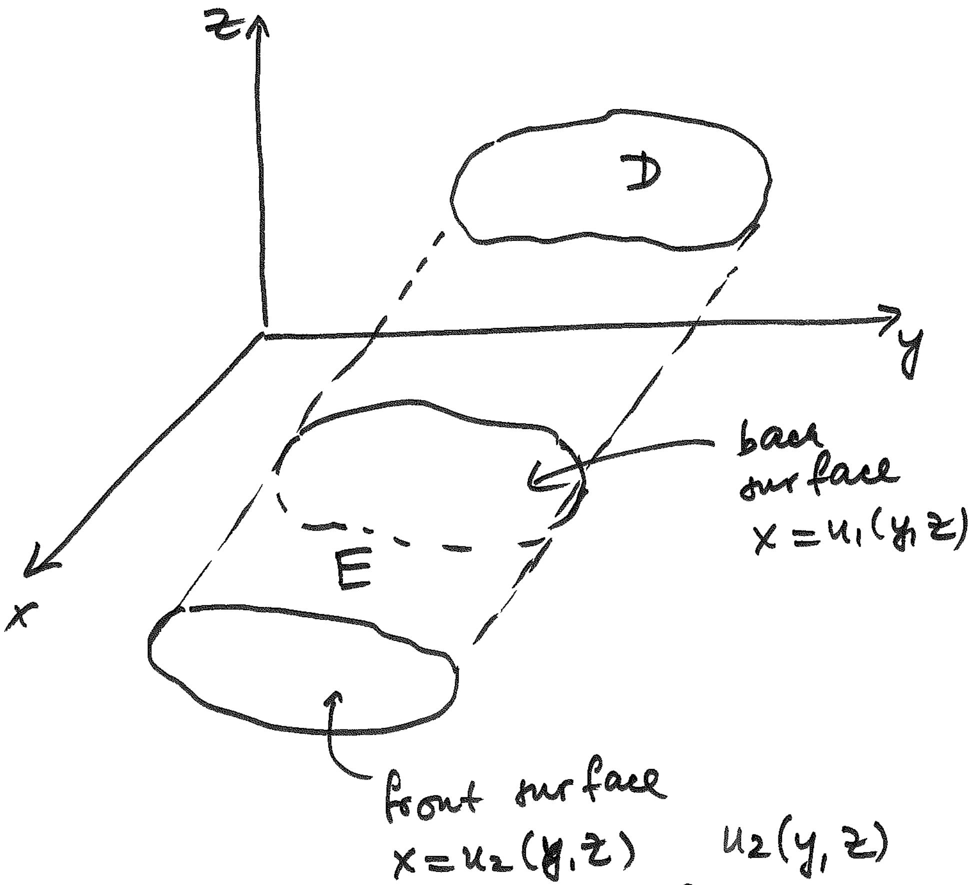
$$= \frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{-x+1} (1-x-y)^2\,dy\,dx = -\frac{1}{2} \int_{x=0}^{1} \frac{(1-x-y)^3}{3} \Big|_{y=0}^{y=-x+1} dx =$$

$$= -\frac{1}{6} \int_{x=0}^{1} \left\{ \cancelto{0}{(1-x-[-x+1])^3} - (1-x-0)^3 \right\} dx = \frac{1}{6} \int_0^1 (1-x)^3\,dx =$$

$$= -\frac{1}{6} \cdot \frac{1}{4} (1-x)^4 \Big|_0^1 = -\frac{1}{24} \cdot (0-1) = \boxed{\frac{1}{24}}$$

a Solid E is of <u>type 2</u> if it can be written as

$$E = \{(x,y,z) : (y,z) \in D,\ \underbrace{u_1(y,z) \le x \le u_2(y,z)}_{\text{continuous}}\}$$



$dA = dy\,dz$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{x=u_1(y,z)}^{u_2(y,z)} \quad dx \right] dA$$

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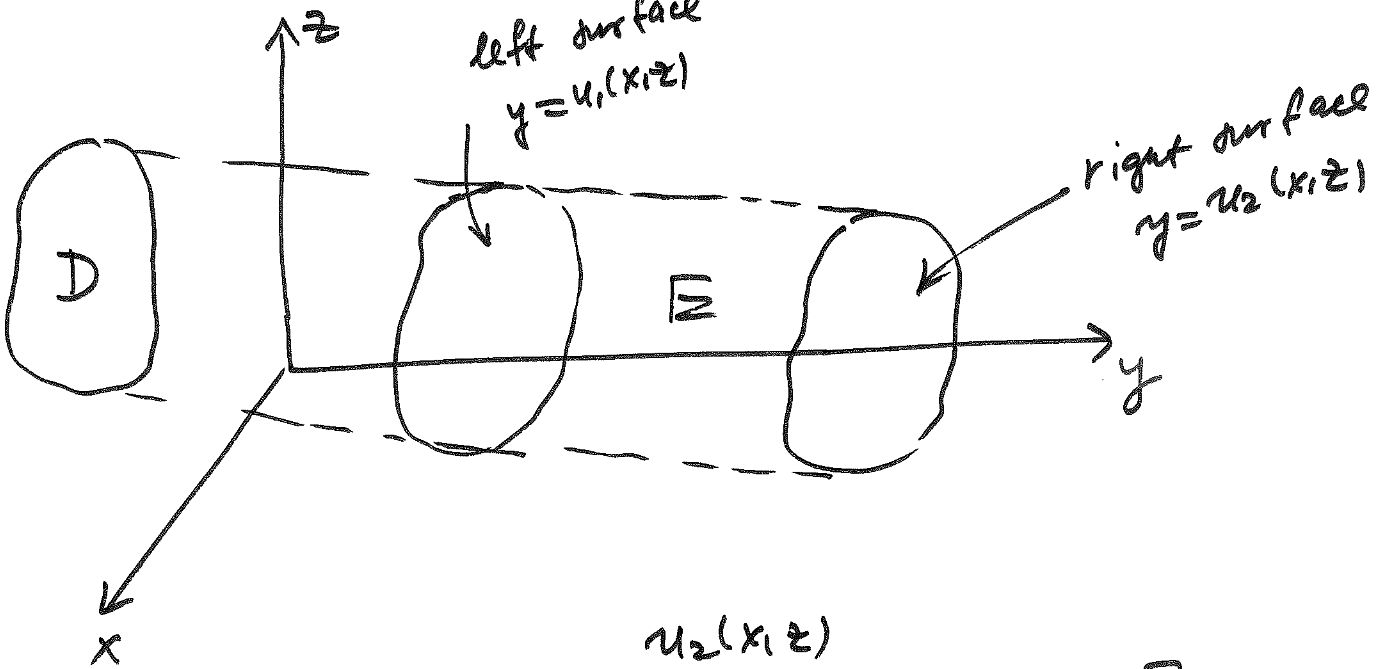
a solid E is of <u>type 3</u> if it can be written as

$E = \{(x, y, z) : (x, z) \in D, \ u_1(x, z) \le y \le u_2(x, z)\}$



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{y=u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

$dA = dx\,dz$