Applications of triple integrals

$f(x,y,z) = 1 \Rightarrow \iiint_E 1\, dV = V(E)$: volume of solid E

Let $\rho(x,y,z)$ be density function at pt (x,y,z). Solid occupies region E and has density $\rho(x,y,z)$.

$m = \iiint_E \rho(x,y,z)\, dV$: total mass of solid

$$M_{yz} = \iiint_E x\, \rho(x,y,z)\, dV$$

$$M_{xz} = \iiint_E y\, \rho(x,y,z)\, dV$$

$$M_{xy} = \iiint_E z\, \rho(x,y,z)\, dV$$

moments about coordinate yz-, xz- and xy-planes

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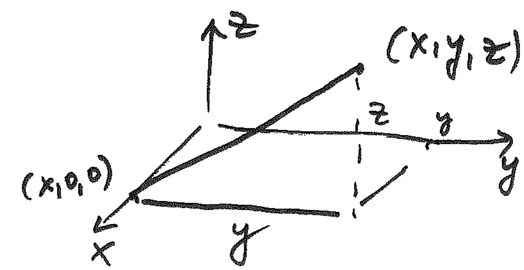
$\bar{x} = \dfrac{M_{yz}}{m}$ $\quad \bar{y} = \dfrac{M_{xz}}{m}$ $\quad \bar{z} = \dfrac{M_{xy}}{m}$: coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the center of mass

$$I_x = \iiint_E \underbrace{(y^2+z^2)}_{(\text{distance})^2 \text{ from } (x,y,z) \text{ to } x\text{-axis}} \rho(x,y,z)\, dV$$



$$I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\, dV$$

moments of inertia about x-, y-, and z-axes

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\, dV$$

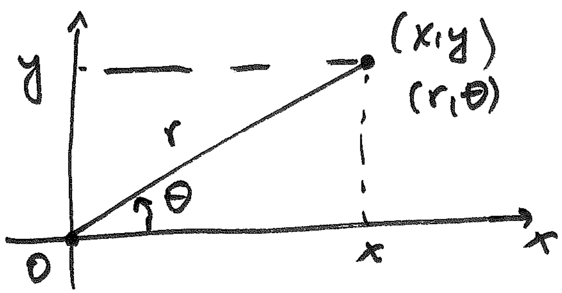
Charge

$\sigma(x,y,z)$: charge density over solid E

$Q = \iiint_E \sigma(x,y,z)\, dV$: total electric charge

15.7 Triple Integrals in Cylindrical Coordinates

Recall **polar coordinates** (r, θ) 2D case



$x = r\cos\theta, \quad y = r\sin\theta, \qquad r \geq 0$

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

$\frac{y}{x} = \tan\theta \qquad 0 \leq \theta < 2\pi$

$\theta = \arctan\frac{y}{x} + \pi$, $x<0, y>0$ $\qquad$ $\theta = \arctan\frac{y}{x}$, $x>0, y>0$

$\theta = \arctan\frac{y}{x} + \pi$, $x<0, y<0$ $\qquad$ $\theta = \arctan\frac{y}{x} + 2\pi$, $x>0, y<0$

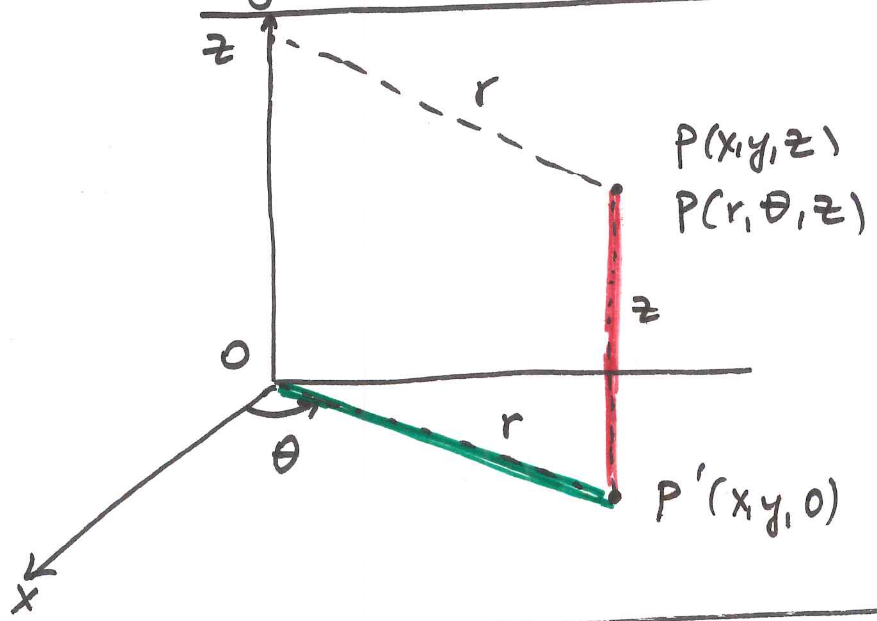
3D case:

cylindrical or spherical coordinates

cylindrical – if body has symmetry about an axis

spherical – if body has symmetry about a point

Cylindrical Coordinates (r, θ, z)



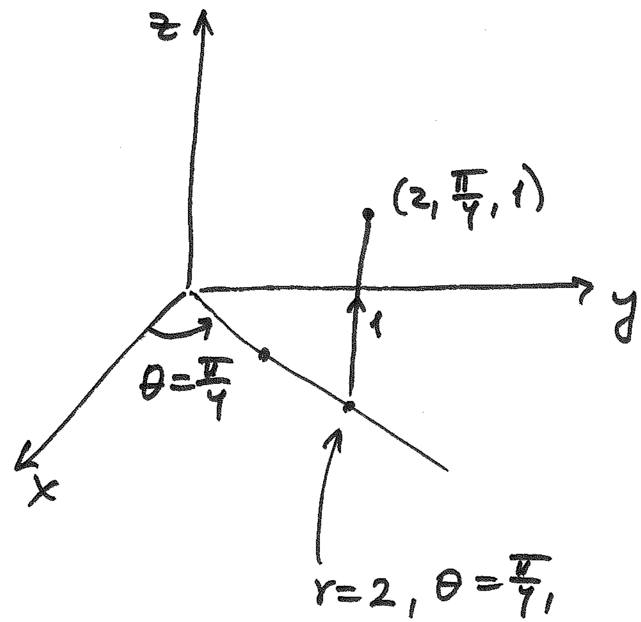
Here (r, θ) are polar coordinates of projection P' of pt P onto xy-plane

$x = r\cos\theta$	$y = r\sin\theta$	$z = z$	$r \geq 0$
$x^2 + y^2 = r^2$	$\frac{y}{x} = \tan\theta$	$z = z$	
	$0 \leq \theta < 2\pi$		

Ex Plot the point whose cylindrical coordinates $(2, \frac{\pi}{4}, 1)$ are given. Then find rectangular / Cartesian coordinates.

$\left(2, \frac{\pi}{4}, 1\right)$ $\quad (r, \theta, z)$

$r = 2, \quad \theta = \frac{\pi}{4}, \quad z = 1$



$x = r\cos\theta = 2\cos\frac{\pi}{4} = 2\,\frac{\sqrt{2}}{2} = \sqrt{2}$

$y = r\sin\theta = 2\sin\frac{\pi}{4} = 2\,\frac{\sqrt{2}}{2} = \sqrt{2}$

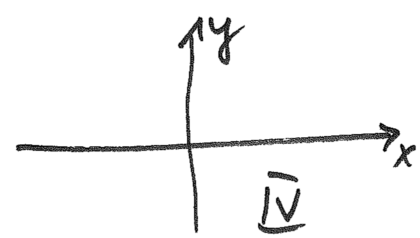
$z = z = 1$

$\therefore\ (\sqrt{2}, \sqrt{2}, 1)$: cartesian coordinates of the given pt

Ex Change from rectangular to cylindrical coordinates :

$(1, -1, 4)$

$x = 1, \quad y = -1, \quad z = 4$



$r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\tan\theta = \frac{y}{x}$ $\qquad x = 1 > 0, \quad y = -1 < 0$

$\Rightarrow (1,-1)$ is in $\overline{\text{IV}}$ quadrant

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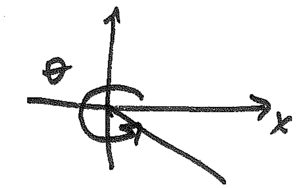
$\therefore\ \theta = \arctan\frac{y}{x} + 2\pi = \arctan\left(\frac{-1}{1}\right) + 2\pi = \arctan(-1) + 2\pi \overset{\text{arctan}}{=}$ is odd function

$\arctan(-t) = -\arctan t$: odd function

$= -\arctan 1 + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4} \in \underline{IV}$ quadrant

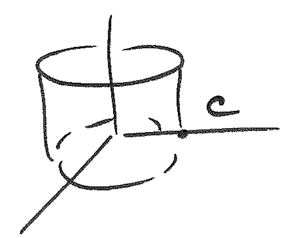


$z = z = 4$

$\therefore\ (\sqrt{2}, \frac{7\pi}{4}, 4)$: cylindrical coordinates of the given pt

Cylindrical coordinates are useful for regions that have symmetry about an axis. Then cylindrical coordinate system may be used where z-axis coincides w/ axis of symmetry.

Ex Circular cylinder $x^2+y^2=c^2$



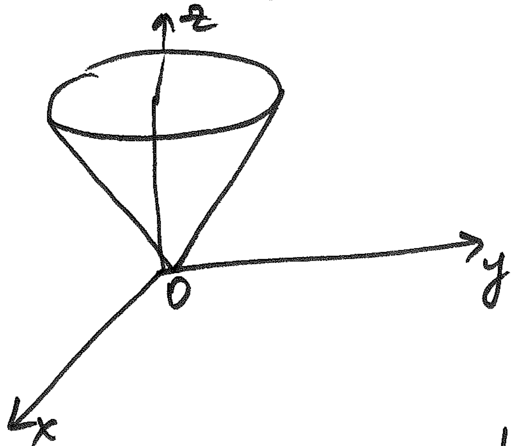
in cylindrical coordinates:

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$\Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = c^2 \Rightarrow r^2 = c^2 \Rightarrow \boxed{r=c}$

Ex $\boxed{z=r}$ $\qquad r \geq 0 \Rightarrow z \geq 0$

$z^2 = r^2 = x^2+y^2 \Rightarrow \boxed{z^2 = x^2+y^2}$



$z = k \Rightarrow x^2+y^2 = k^2$: circle w/ center at origin and rad k

as $z \nearrow$, $k \nearrow$ (radius also $\nearrow$)

$\Rightarrow$ surface is upper half of the cone

Evaluating triple integrals w/ cylindrical coordinates

E : region of type 1

$E = \{(x,y,z) : (x,y) \in D, \; u_1(x,y) \leq z \leq u_2(x,y)\}$

D: projection of E onto xy-plane

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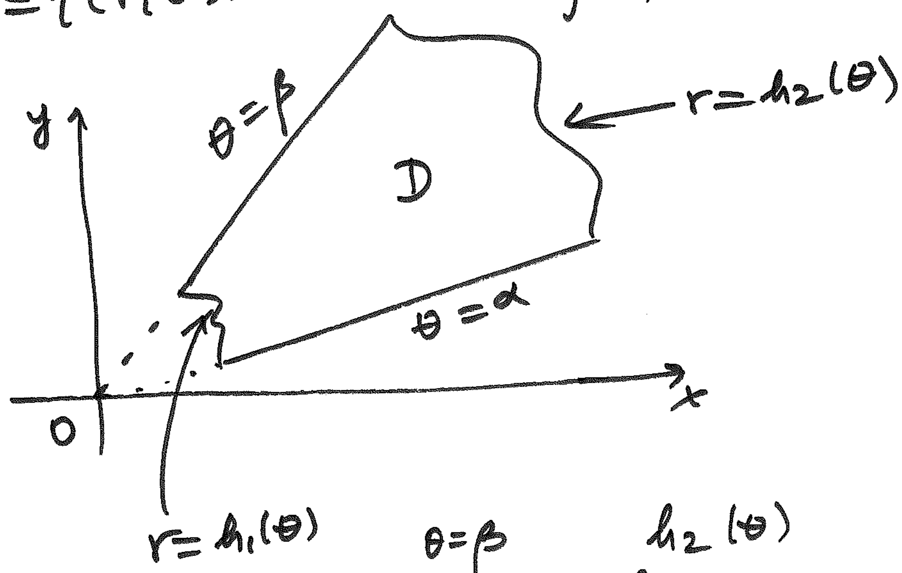
D: in polar coordinates

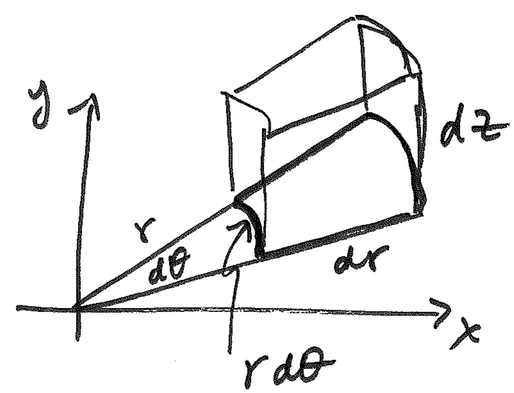
$D = \{(r,\theta):\quad \alpha \le \theta \le \beta,\quad h_1(\theta) \le r \le h_2(\theta)\}$



$$\iiint_E f(x,y,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{h_2(\theta)} \int_{z=u_1(r\cos\theta,\, r\sin\theta)}^{z=u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\; dz \cdot \underline{r}\,dr\,d\theta$$

!

$dV = dx\,dy\,dz$

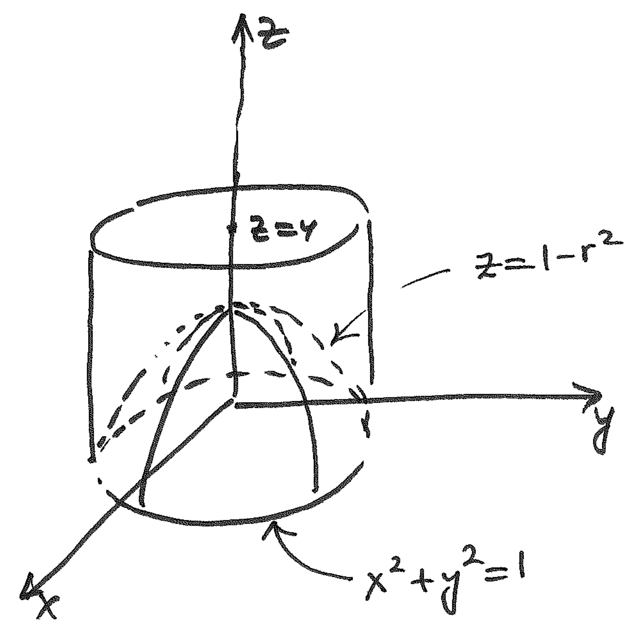


$dV = dz \cdot \underbrace{r\,dr\,d\theta}_{dA}$

$dV = r\,dr\,d\theta\,dz$

Ex Solid E lies within cylinder $x^2+y^2=1$, below plane $z=4$ and above paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

$$m = \iiint_E \rho(x,y,z)\, dV \ : \ \text{total mass}$$



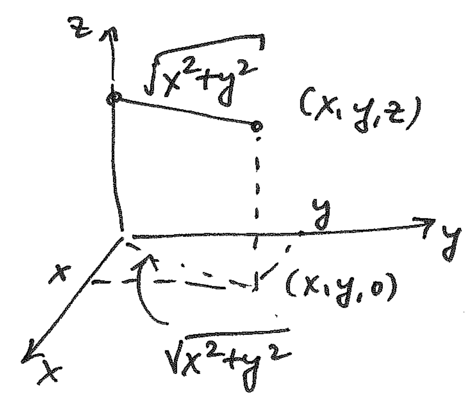
Intersection:

$x^2+y^2=1$ & $z=1-x^2-y^2 \Rightarrow z=0$

when $z=0 \Rightarrow x^2+y^2=1$

$\rho(x,y,z) = k\cdot\sqrt{x^2+y^2} = k\cdot r$

const — distance from z-axis to pt (x,y,z)



$x^2+y^2=r^2$

Cylinder: $x^2+y^2=1$ or $r^2=1 \Rightarrow r=1$

Paraboloid: $z=1-x^2-y^2$ or $z=1-r^2$

$E=\{(r,\theta,z):\ 0\le r\le 1,\ 0\le\theta\le 2\pi,\ 1-r^2\le z\le 4\}$

$$m=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=1-r^2}^{4} \underbrace{kr}_{f(r,\theta,z)}\cdot r\,dz\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=1-r^2}^{4} kr^2\,dz\,dr\,d\theta =$$

$$=\int_{\theta=0}^{2\pi} d\theta\cdot\int_{r=0}^{1} kr^2 z\Big|_{z=1-r^2}^{z=4}\,dr = 2\pi\cdot\int_{r=0}^{1} kr^2\big(\underbrace{4-[1-r^2]}_{3+r^2}\big)\,dr =$$

$$=2\pi k\int_0^1 (3r^2+r^4)\,dr = 2\pi k\cdot\left(r^3+\frac{r^5}{5}\right)\Big|_0^1 = 2\pi k\,\big(\underbrace{1+\tfrac{1}{5}}_{\frac{6}{5}}\big) = \boxed{\frac{12\pi k}{5}}$$