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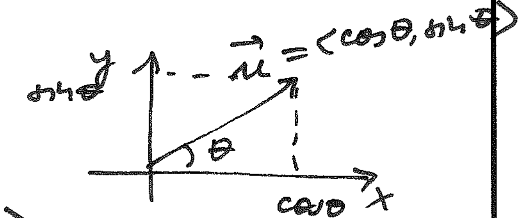
Lecture 36

Review #3

Possible problems:

- compute directional derivatives given by a direction vector $\vec{u}$ or by angle θ that defines direction. Need to know gradient of a function.

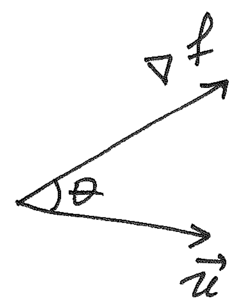


$D_{\vec{u}} f = \nabla f \cdot \vec{u}$ $\quad \vec{u}$: unit vector

if $\vec{u}$ is given by angle θ, then $\vec{u} = \langle \cos\theta, \sin\theta \rangle$

$\nabla f = \langle f_x, f_y \rangle$

Note $\nabla f \cdot \vec{u} = |\nabla f| \cdot \underbrace{|\vec{u}|}_{=1} \cdot \cos\theta = |\nabla f| \cdot \cos\theta$. In this case θ is the angle between ∇f and $\vec{u}$



Note different meaning of θ here and above

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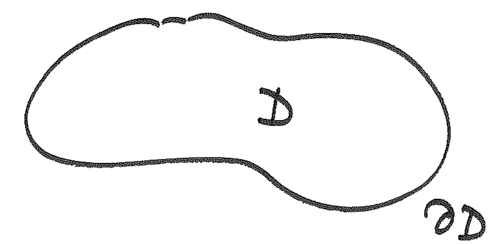

- max / min of a function in D including boundary ∂D:
  - critical points in D: $f_x = 0$, $f_y = 0$
    $\Rightarrow (x_0, y_0)$ or more points or no points

- 2^{nd} derivative test

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(x_0, y_0)}$$

if $D > 0$ and $f_{xx}(x_0, y_0) > 0 \Rightarrow (x_0, y_0)$ is a pt of local min

if $D > 0$ and $f_{xx}(x_0, y_0) < 0 \Rightarrow (x_0, y_0)$ is a pt of local max

if $D < 0 \Rightarrow (x_0, y_0)$ is a saddle point

if $D = 0$: test is inconclusive

On the boundary, we can either parametrize boundary as a function of one variable and also find crit. points or use the method of Lagrange multipliers.

- exam may include a problem on the method of Lagrange multipliers w/ one constraints

$$f(x,y) \longrightarrow \min/\max$$

$+$ constraints $g(x,y)=k$

Solve for λ, x, y:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=k \end{cases} \quad \text{or} \quad \left.\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{array}\right\} \text{solve for } x, y, \lambda$$

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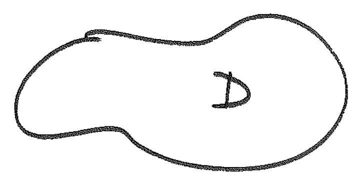
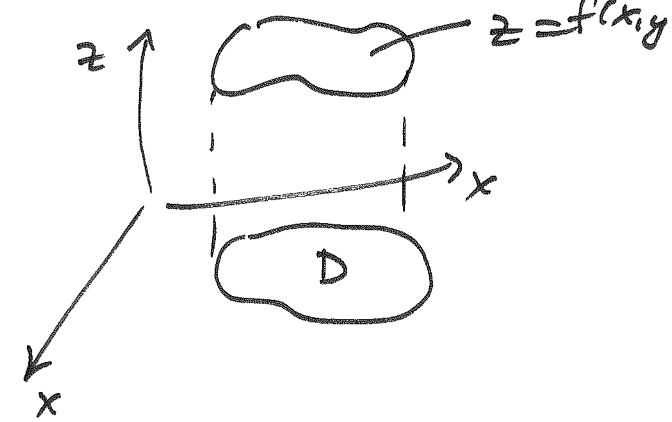
Then evaluate f at all points (crit. points that have local min, max, solutions (points) on the boundary) and pick f_{max}, f_{min} to be global/absolute max and min of f, respectively.

- double integrals:
  - set up a double integral (domain D as of type I or II, limits of integration)
  - evaluate a double integral (iterated integrals, partial integration)
  - area of D:

$$A(D) = \iint_D dA$$

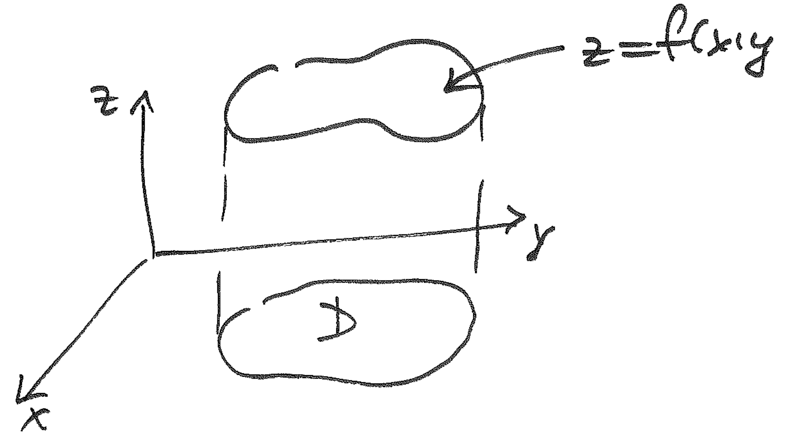


- volume of a solid bounded by surface $z=f(x,y)$ and region D



$$V=\iint_D f(x,y)\,dA$$

- double integrals in polar coordinates



- surface area

$$A(S)=\iint_D \sqrt{1+f_x^2+f_y^2}\,dA$$

- applications (e.g. mass)

Ex Find abs. max and abs. min of

$$f(x,y) = x^2 + 4y^2 + 1$$

over the region $R = \{(x,y) : x^2 + 4y^2 \le 1\}$. Use Lagrange multipliers to check for extreme points on the boundary.

Solution

First we need crit. pts.

$f_x = 2x, \quad f_y = 8y$

$f_x = f_y = 0 \Rightarrow x = y = 0 \Rightarrow (0,0)$ is a crit. pt

2nd derivative test:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}_{(0,0)} = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix} = 16 > 0$$

$D > 0$ & $f_{xx}(0,0) = 2 > 0 \Rightarrow (0,0)$ is a pt of local min

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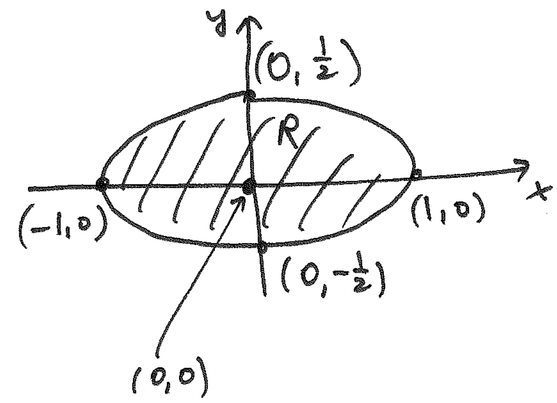
$R = \{(x,y) : x^2 + 4y^2 \le 1\}$

$$x^2 + \frac{y^2}{\left(\frac{1}{2}\right)^2} \le 1$$

$(0,0) \in R$ ✓



Boundary $x^2 + 4y^2 = 1$: ellipse

The method of Lagrange multipliers:

$$f(x,y) = x^2 + 4y^2 + 1 \longrightarrow \text{min/max}$$

subject to a constraint

$$\underbrace{x^2 + 4y^2}_{g(x,y)} = \underbrace{1}_{=k}$$

$g_x = 2x, \quad g_y = 8y$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases} \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda f_y \\ g = 1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda \cdot 2x \\ 8y = \lambda \cdot 8y \\ x^2 + 4y^2 = 1 \end{cases}$$

$2x = \lambda \cdot 2x$

$2x(1-\lambda) = 0 \Rightarrow x = 0 \quad \text{or} \quad \lambda = 1$

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(1) Let $x=0$. Substitute into $\underset{0}{x^2} + 4y^2 = 1 \Rightarrow 4y^2 = 1 \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$\Rightarrow$ points $(0, \frac{1}{2})$, $(0, -\frac{1}{2})$

(2) $\lambda = 1$

$8y = \lambda \cdot 8y \Rightarrow 8y = 8y$ no info

$2x = \lambda \cdot 2x \Rightarrow 2x = 2x$ no info

(3) $8y = \lambda \cdot 8y \Rightarrow 8y(1-\lambda) = 0 \Rightarrow y = 0$ or $\lambda = 1$ — no info, we saw in (2)

Let $y=0 \Rightarrow x^2 + \cancel{4y^2} = 1 \Rightarrow x^2 = 1$ or $x = \pm 1 \Rightarrow (1,0)$ & $(-1,0)$

So we have 5 points:

$(0,0)$ $(0, \frac{1}{2})$ $(0, -\frac{1}{2})$ $(1,0)$ $(-1,0)$

Evaluate $f(x,y) = x^2 + 4y^2 + 1$ at these points:

$f(0,0) = \boxed{1}$

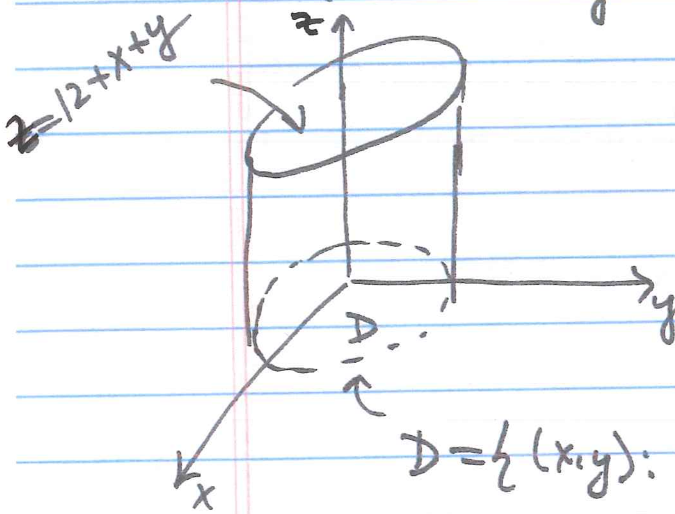
$f(0, \frac{1}{2}) = 0 + 1 + 1 = \boxed{2} = f(0, -\frac{1}{2})$

$f(1,0) = f(-1,0) = 1 + 0 + 1 = \boxed{2}$

$\therefore f_{\text{abs min}} = 1$ $f_{\text{abs max}} = 2$

Ex Use double integral to evaluate the volume of the given region:

the segment of the cylinder $x^2+y^2=1$ bounded above by the plane $z=12+x+y$ and below by $z=0$.

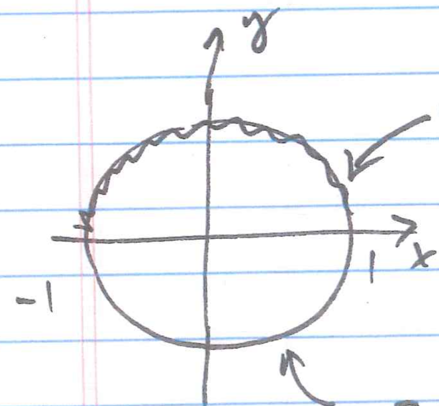


$$V = \iint_D f(x,y)\,dA \;=$$

$f \geq 0$

$z = f(x,y)$

$D = \{(x,y):\ x^2+y^2 \leq 1\}$



$y=\sqrt{1-x^2}$

$y=-\sqrt{1-x^2}$

D: domain of type I

$D = \{(x,y):\ -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{(12+x+y)}_{f(x,y)}\,dy\,dx$$

We could also use polar coordinates:

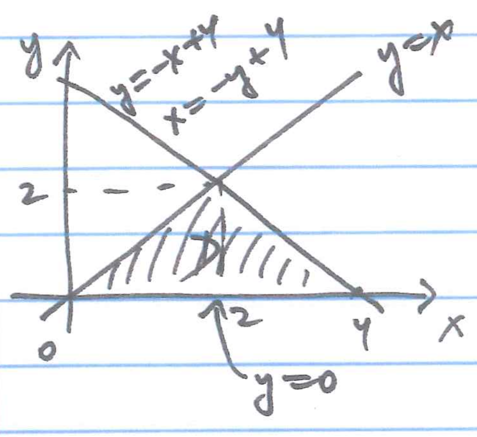
$$x = r\cos\theta, \quad y = r\sin\theta$$

$$D = \{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$$

$$\Rightarrow V = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (12 + \underbrace{r\cos\theta}_{x} + \underbrace{r\sin\theta}_{y})\, r\,dr\,d\theta = \ldots$$

easier to evaluate than using cartesian coordinates

Ex Let D be region bounded by the lines $y=x$, $x+y=4$ and $y=0$. Evaluate $\iint_D y\,dA$ and sketch D.



$$x+y=4 \Rightarrow y = -x+4$$
$$\Downarrow$$
$$x = 4-y$$

Treat D as domain of type II:

$$D = \{(x,y) : 0 \le y \le 2, \quad y \le x \le -y+4\}$$

$$\Rightarrow \iint_D y\,dA = \int_0^2 \int_y^{-y+4} y\,dx\,dy =$$

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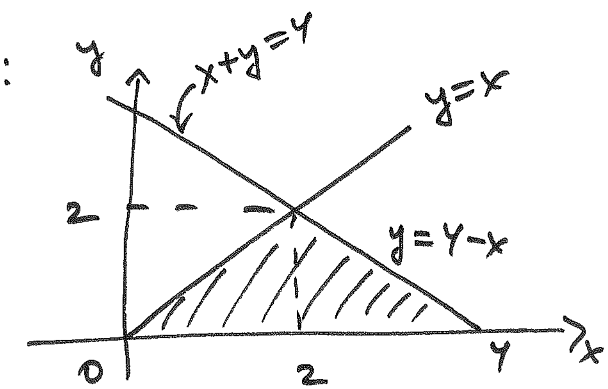
$$\iint_D y\,dA = \int_{y=0}^{2}\int_{x=y}^{-y+4} y\,dx\,dy = \int_{y=0}^{2} y \int_{x=y}^{-y+4} dx\,dy =$$

$$= \int_{y=0}^{2} y \; x\Big|_{x=y}^{-y+4} dy = \int_0^2 y\,(\underbrace{-y+4-y}_{-2y+4})\,dy = \int_0^2 \left(-2y^2+4y\right) dy =$$

$$= \left(-\frac{2}{3}y^3 + 2y^2\right)\Big|_0^2 = -\frac{2}{3}\cdot\underbrace{2^3}_{8} + \underbrace{2\cdot 4}_{8} = 8\left(1-\frac{2}{3}\right) = 8\cdot\frac{1}{3} = \boxed{\frac{8}{3}}$$

We can also regard domain D as of type I:



$$\iint_D y\,dA = \int_{x=0}^{2}\int_{y=0}^{x} y\,dy\,dx + \int_{x=2}^{4}\int_{y=0}^{4-x} y\,dy\,dx =$$

$$= \ldots$$