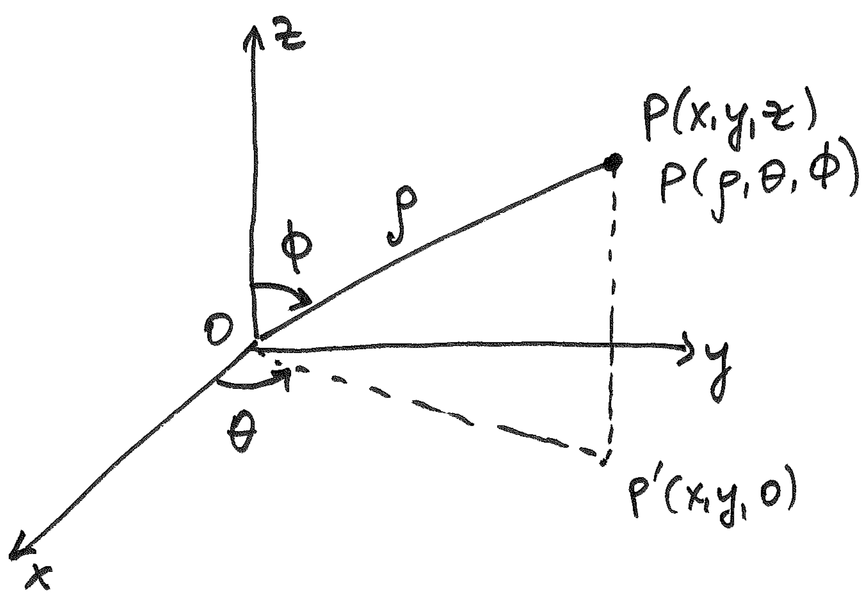
Lecture 37

15.8 Spherical Coordinates in Triple Integrals

Spherical coordinates are used to parametrize objects that have symmetry wrt a point: spheres and cones.



Spherical coordinates

ρ: distance from O to P

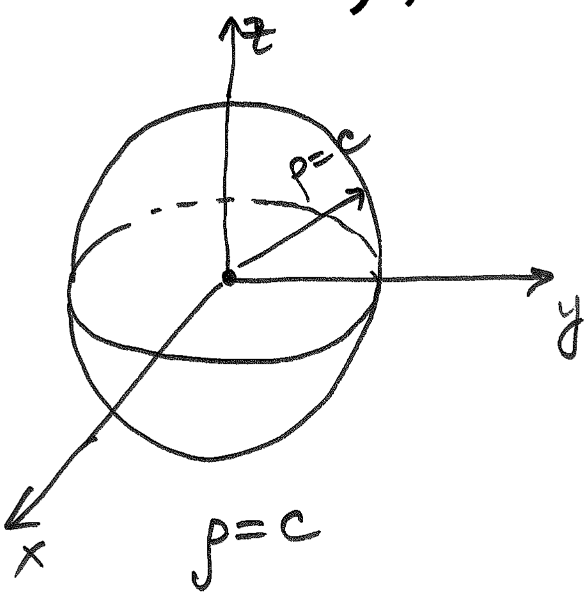
$\rho \geq 0$

θ: the same angle as in cylindrical/polar coordinates

$0 \leq \theta \leq 2\pi$

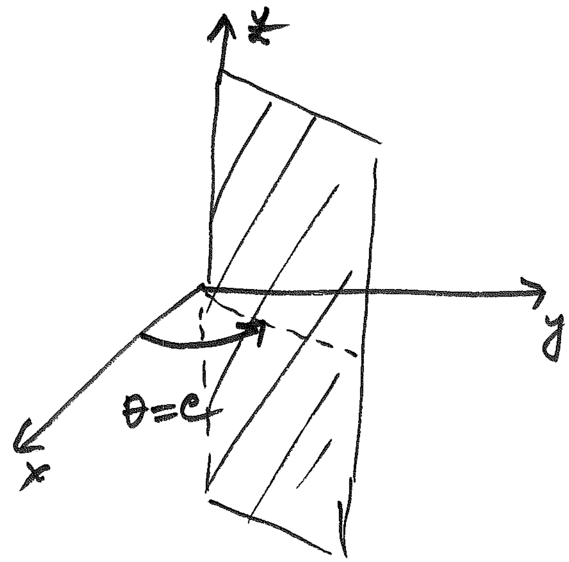
ϕ: angle measured from positive z-axis to OP

$0 \leq \phi \leq \pi$

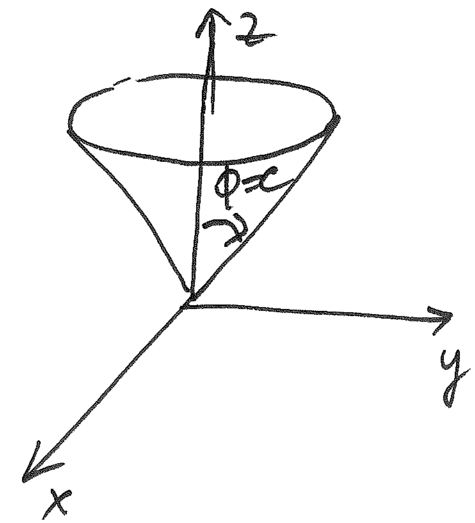


$\rho = c$

sphere centered at origin w/ radius c

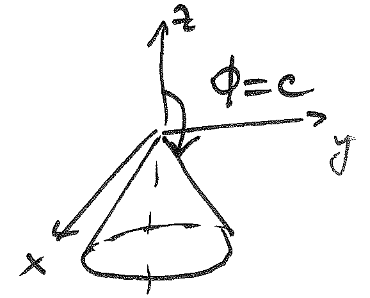


Plane $\theta = c$



upper half of cone if $0 < \phi < \frac{\pi}{2}$

lower half of cone if $\frac{\pi}{2} < \phi < \pi$



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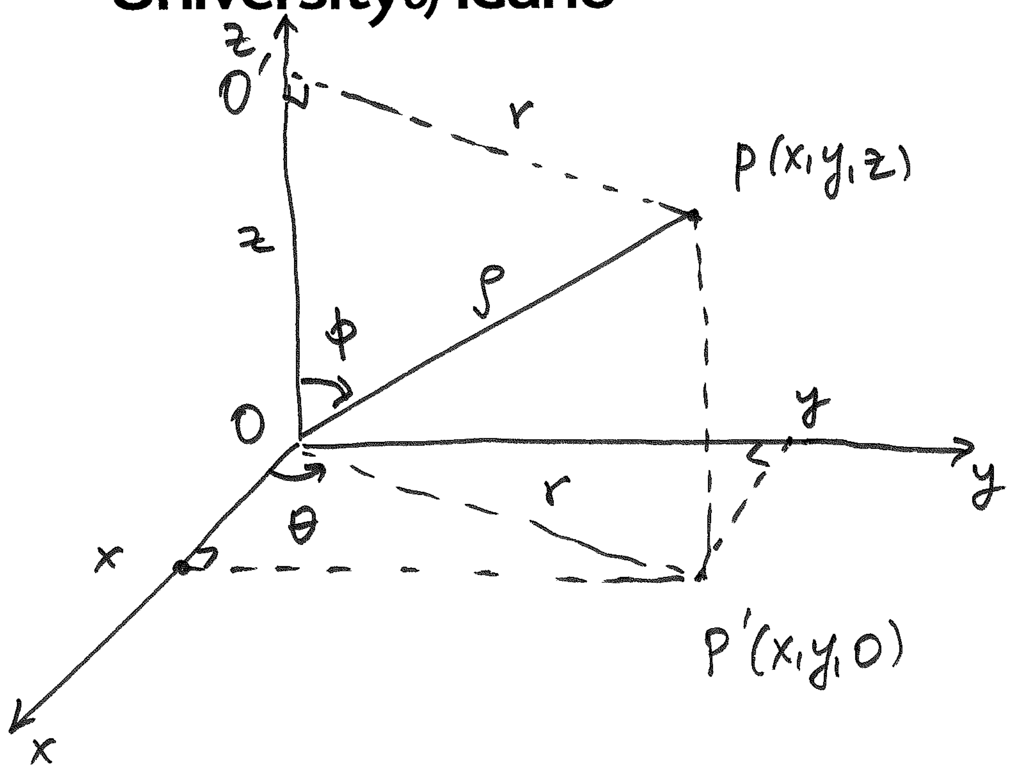

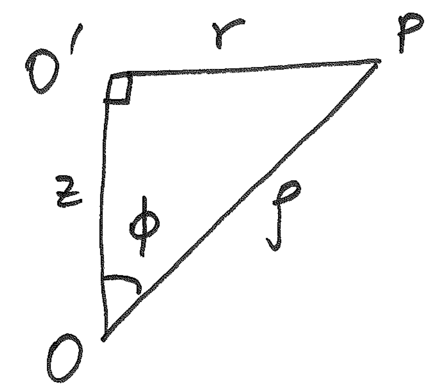

From $\Delta_{OO'P}$:

$z = \rho \cos \phi$

$r = \rho \sin \phi$

$x = r \cos \theta \qquad y = r \sin \theta$

but $\quad r = \rho \sin \phi$

$\therefore$

$$\boxed{\begin{array}{lll} x = \rho \sin \phi \cos \theta & y = \rho \sin \phi \sin \theta & z = \rho \cos \phi \\ \rho \geq 0 & 0 \leq \theta \leq 2\pi & 0 \leq \phi \leq \pi \end{array}}$$

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We can see / know:

$$x^2+y^2=r^2$$

From $\triangle OO'P$: $\rho^2 = r^2+z^2$

$$\therefore \quad \boxed{\rho^2 = x^2+y^2+z^2}$$

$\boxed{\tan\theta = \frac{y}{x}}$ as in cylindrical / polar coordinates

$$z=\rho\cos\phi \rightarrow \boxed{\cos\phi = \frac{z}{\rho}} \Rightarrow \begin{cases} \phi = \arccos\frac{z}{\rho} & \text{if } 0<\phi<\frac{\pi}{2} \\ \phi = \frac{\pi}{2} - \arcsin\frac{z}{\rho} & \text{if } \frac{\pi}{2}<\phi<\pi \end{cases}$$

Ex Plot point whose spherical coordinates are given.
Then find rectangular coordinates.

$$\left(2, \frac{\pi}{3}, \frac{\pi}{4}\right) \qquad (\rho, \theta, \phi)$$

$$\rho = 2, \quad \theta = \frac{\pi}{3}, \quad \phi = \frac{\pi}{4}$$

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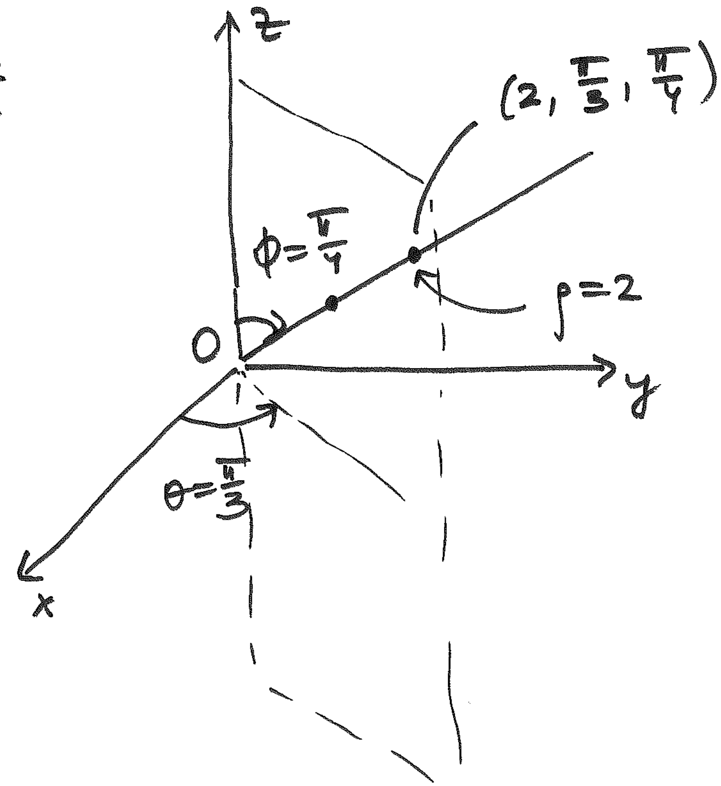
$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{4} \cos\frac{\pi}{3} = 2\,\frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}}{2}$

$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{4} \sin\frac{\pi}{3} = 2\,\frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$

$z = \rho \cos\phi = 2 \cos\frac{\pi}{4} = 2\,\frac{\sqrt{2}}{2} = \sqrt{2}$

$\therefore \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$: cartesian/rectangular coordinates of the given point



Ex Change from rectangular to spherical coordinates.

$(-1, 1, \sqrt{6})$ $\qquad (x, y, z)$

$\quad x \quad\; y \quad\; z$

Solution:

$\rho^2 = x^2 + y^2 + z^2 \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2 + 1^2 + (\sqrt{6})^2} = \sqrt{8}$
$= 2\sqrt{2}$

$\boxed{\rho = 2\sqrt{2}}$

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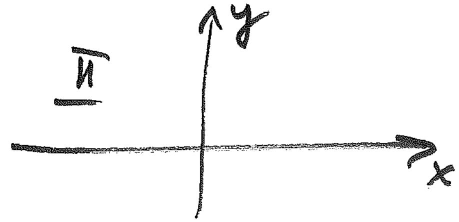
θ is the same as in polar/cylindrical coordinates

$$\tan\theta = \frac{y}{x}$$

$x = -1 < 0$, $y = 1 > 0$



II quadrant $\Rightarrow$

$$\theta = \arctan\frac{y}{x} + \pi = \arctan\left(\frac{1}{-1}\right) + \pi \underset{\substack{\text{is odd} \\ \text{function}}}{\overset{\arctan}{=}} -\arctan 1 + \pi =$$

$$= -\frac{\pi}{4} + \pi = \boxed{\frac{3\pi}{4} = \theta}$$

$$\cos\phi = \frac{z}{\rho}$$

$z = \sqrt{6} > 0 \Rightarrow 0 < \phi < \frac{\pi}{2} \Rightarrow \phi = \arccos\frac{z}{\rho} = \arccos\frac{\sqrt{6}}{2\sqrt{2}} =$

$$= \arccos\frac{\sqrt{2}\cdot\sqrt{3}}{2\sqrt{2}} = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$$\therefore \boxed{\phi = \frac{\pi}{6}}$$

$\therefore \left(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6}\right)$: spherical coordinates

Ex   Case $\cos\phi < 0$ $\frac{z}{\rho}$

Let $\cos\phi = -\frac{1}{2} \Rightarrow \frac{\pi}{2} < \phi < \pi$

$$\phi = \frac{\pi}{2} - \arcsin\left(-\tfrac{1}{2}\right) \overset{\text{arcsin is}}{\underset{\text{odd } f^{n}}{=}} \frac{\pi}{2} + \arcsin\frac{1}{2} = \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

Evaluating triple integrals with spherical coordinates

Spherical wedge:

$$E = \{(\rho, \theta, \phi) : \quad a \le \rho \le b, \quad \alpha \le \theta \le \beta, \quad c \le \phi \le d\}$$

$$a \geq 0 \qquad \beta - \alpha \le 2\pi \qquad d - c \le \pi$$

This domain E is between spheres $\rho = a$ and $\rho = b$, planes $\theta = \alpha$, $\theta = \beta$, and half-cones $\phi = c$, $\phi = d$.

Divide E into small sub-wedges by spheres $\rho = \rho_i$, vertical planes $\theta = \theta_j$ and half-cones $\phi = \phi_k$.