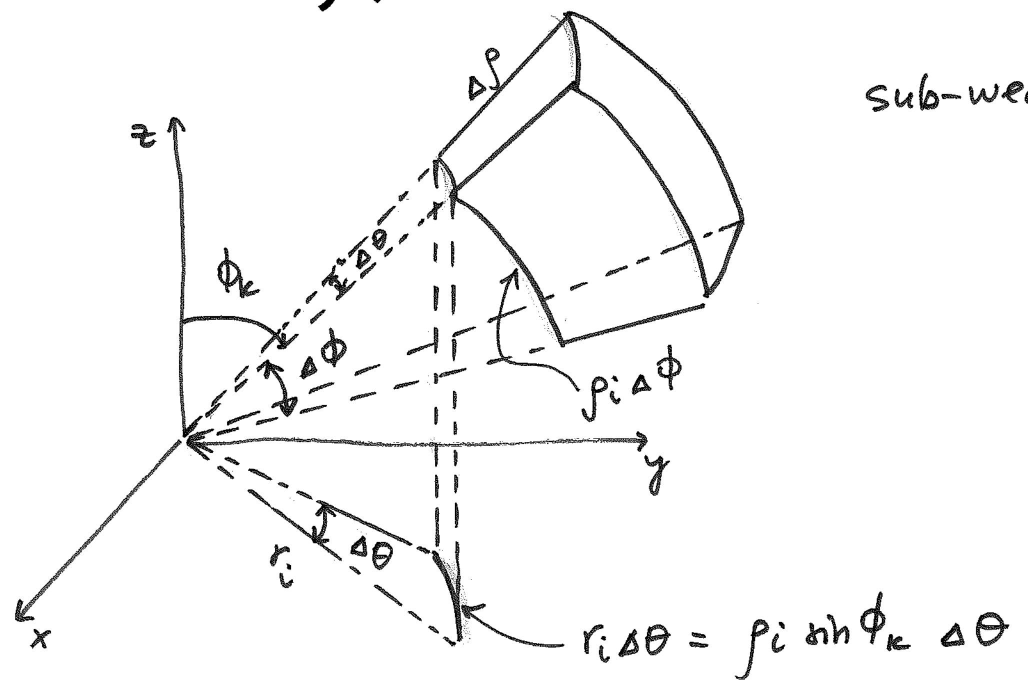
Lecture 38



sub-wedge E_{ijk}

$r_i = \rho_i \sin\phi_k$

Volume of $E_{ijk} \approx \Delta\rho \cdot (\rho_i \sin\phi_k \Delta\theta)\cdot(\rho_i \Delta\phi)$

By Mean Value Thm, we can find sample points (values ρ_i^*, ϕ_k^*):

$$\text{Volume of } E_{ijk} = (\rho_i^*)^2 \sin\phi_k^* \,\Delta\rho\,\Delta\theta\,\Delta\phi$$

Adding contributions from all spherical wedges and taking limit as partitions of ρ, θ, ϕ tend to ∞, we get

$$\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V_{ijk}$$

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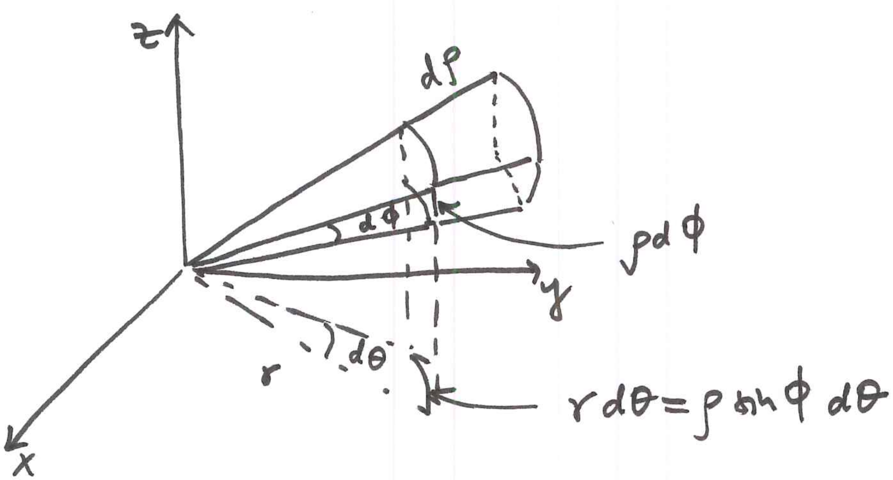
$$= \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\rho_i^* \sin\phi_k^* \cos\theta_j, \ \rho_i^* \sin\phi_k^* \sin\theta_j^*, \ \rho_i^* \cos\phi_j^*) \cdot (\rho_i^*)^2 \sin\phi_k^* \, \Delta\rho \, \Delta\theta \, \Delta\phi$$

$\therefore$

$$\boxed{\iiint_E f(x,y,z)\, dV = \int_{\phi=c}^{d} \int_{\theta=\alpha}^{\beta} \int_{\rho=a}^{b} f(\rho\sin\phi\cos\theta, \ \rho\sin\phi\sin\theta, \ \rho\cos\theta) \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$



$$dV = d\rho \cdot \rho\, d\phi \cdot \rho \sin\phi \, d\theta$$

$$dV = \underbrace{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

volume element in spherical coordinates

<u>Note</u> We can use spherical coordinates when a solid is bounded by spheres and cones.

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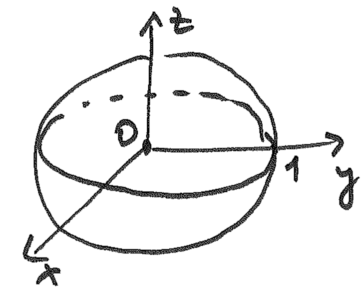
Ex Evaluate

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV =$$



B: unit ball $\quad B = \{(x,y,z): \ x^2+y^2+z^2 \le 1\}$

Solution:

In spherical coordinates

$$B = \{(\rho, \theta, \phi): \ 0 \le \rho \le 1, \ \ 0 \le \theta \le 2\pi, \ \ 0 \le \phi \le \pi\}$$

$$f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}} = e^{(\rho^2)^{3/2}} = e^{\rho^{2 \cdot \frac{3}{2}}} = e^{\rho^3}$$

$$= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} e^{\rho^3} \cdot \underbrace{\rho^2 \sin\phi} \cdot d\rho \, d\theta \, d\phi =$$

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{\phi=0}^{\pi} \sin\phi \, d\phi \cdot \int_{\rho=0}^{1} e^{\rho^3} \cdot \rho^2 \, d\rho \quad \boxed{=}$$

product of 3 single integrals

$$\int_{\theta=0}^{2\pi} d\theta = 2\pi$$

$$\int_{\phi=0}^{\pi} \sin\phi \, d\phi = -\cos\phi \Big|_0^{\pi} = -(\underbrace{\cos\pi}_{=-1} - \underbrace{\cos 0}_{=1}) = 2$$

$$\int_{\rho=0}^{1} e^{\rho^3} \rho^2 \, d\rho = \left| \begin{array}{l} u = \rho^3 \\ du = 3\rho^2 d\rho \\ \rho=0 \Rightarrow u=0 \\ \rho=1 \Rightarrow u=1 \end{array} \right| = \int_0^1 e^u \frac{1}{3} du = \frac{1}{3} e^u \Big|_0^1 = \frac{1}{3}(e-1)$$

$$\boxed{=} \; 2\pi \cdot 2 \cdot \frac{1}{3}(e-1) = \boxed{\frac{4\pi}{3}(e-1)}$$

Ex Use spherical coordinates to find volume of a solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.

Volume $V=\iiint_E dV$ E : solid

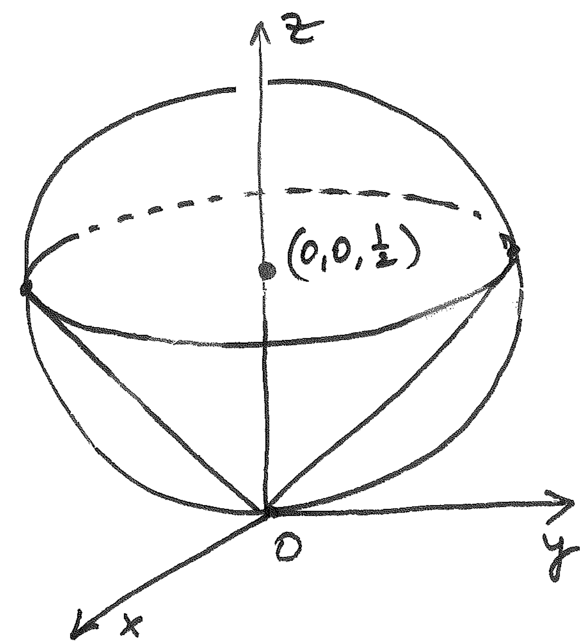
Cone: $z=\sqrt{x^2+y^2}\geq 0 \Rightarrow z\geq 0$: upper half of the cone

Sphere: $x^2+y^2+z^2=z$

complete square: $x^2+y^2+z^2-z=0$

$x^2+y^2+z^2-2\cdot\frac{1}{2}z+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2=0$

$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2$: sphere centered at $(0,0,\frac{1}{2})$ w/ radius $r=\frac{1}{2}$



Intersection:

$z=\sqrt{x^2+y^2}$: cone $\qquad \underbrace{x^2+y^2}_{= z^2}+z^2=z$: sphere

or $z^2=x^2+y^2$ $\qquad \Rightarrow 2z^2=z$

or $z(1-2z)=0$

$z_1=0 \quad z_1=\frac{1}{2}$: surfaces intersect at $z=0$ (origin) and a circle $x^2+y^2=1$ at $z=\frac{1}{2}$

Cone: $z=\sqrt{x^2+y^2}$

Recall

$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$: spherical coordinates

$$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}=\sqrt{\rho^2\sin^2\phi}=\rho\sin\phi$$

$\cos^2\theta+\sin^2\theta=1$

$\rho^2\sin^2\phi(\underbrace{\cos^2\theta+\sin^2\theta}_{=1})$

$\sin\phi>0$ since we have upper half cone

$\Rightarrow \rho\cos\phi = \rho\sin\phi \;\Big|\; \frac{1}{\rho} \Rightarrow \cos\phi = \sin\phi \Rightarrow \boxed{\phi = \frac{\pi}{4}}$ eq^n of the upper half cone

Sphere:

$\underbrace{x^2+y^2+z^2}_{\rho^2} = \underbrace{z}_{\rho\cos\phi} \Rightarrow \rho^2 = \rho\cos\phi \Rightarrow \boxed{\rho = \cos\phi}$ eq^n of the sphere

Hence,

$E = \{(\rho,\theta,\phi):\ 0 \le \rho \le \cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4}\}$

$$V(E) = \iiint_E dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \ldots = \frac{\pi}{8}$$

15.9 Change of variables in multiple integrals

Recall

$$\int_a^b f(x)\,dx = \left| \begin{array}{l} x = g(u) \\ dx = g'(u)\,du \\ x = a \Rightarrow a = g(c) \\ x = b \Rightarrow b = g(d) \end{array} \right| = \int_c^d f(g(u))\, g'(u)\,du = \int_c^d f(x(u))\, \underline{\underline{\frac{dx}{du}}}\, du$$

<u>Ex</u> Double integral. Polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\, \overbrace{\underline{\underline{r}}\,dr\,d\theta}^{dA}$$

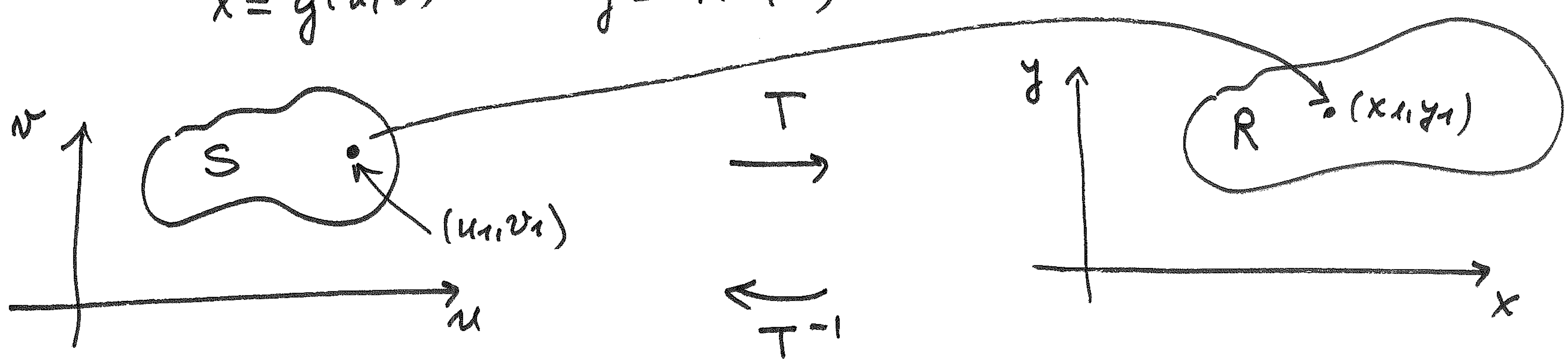
R – region in xy-plane

S – region in $r\theta$-plane

More generally, consider a change of variables

$$T(u,v) = (x,y)$$

where $x = g(u,v)$ $\quad y = h(u,v)$



We say that (x_1, y_1) is an image of (u_1, v_1).

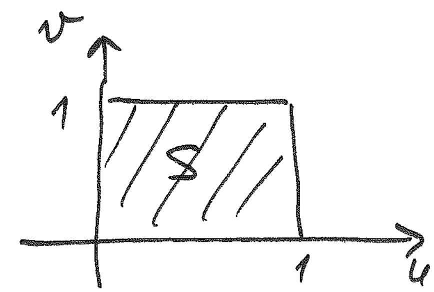
<u>Def</u> If no two different points have the same image, then T is <u>one-to-one</u> transformation.

<u>Def</u> If transformation T is one-to-one, then there exists an <u>inverse</u> tranformation T^{-1} that maps region R into region S.

$u = G(x, y) \qquad v = H(x, y)$

<u>Ex</u> $x = u^2 - v^2, \quad y = 2uv$: transformation T

$S = \{(u, v) : 0 \le u \le 1, \ 0 \le v \le 1\}$: square



Find image of S