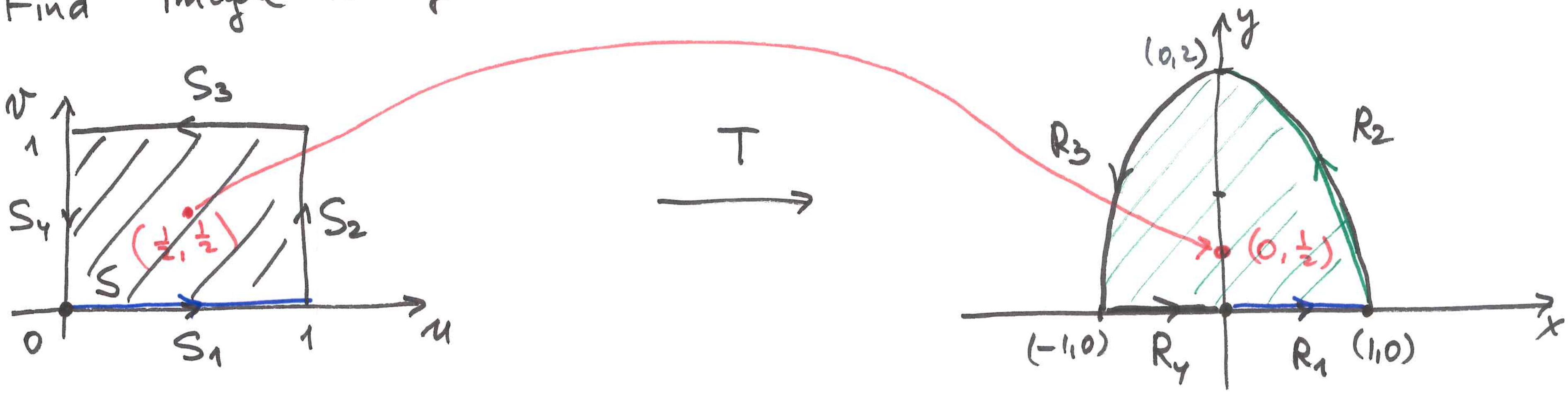
Lecture 39

<u>Ex</u> $x = u^2 - v^2, \quad y = 2uv \ : \quad T$

$S = \{(u,v) : 0 \le u \le 1, \ 0 \le v \le 1\}$: square

Find image R of S.



<u>Note</u> Boundary of S goes into boundary of R.

$S_1 : v = 0, \quad 0 \le u \le 1 \Rightarrow x = u^2 - \cancel{v^2}^{\,0} = u^2, \quad y = 2u\cancel{v}^{\,0} = 0$

$\therefore \ x = u^2, \ y = 0$

$0 \le u \le 1 \Rightarrow 0 \le x \le 1$

$u = 0, v = 0 \Rightarrow x = 0, y = 0 \Rightarrow (0,0) \rightarrow (0,0)$

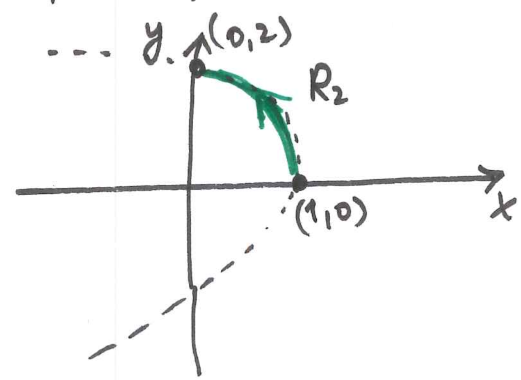
in uv-plane, in xy-plane

$u = 1, v = 0 \Rightarrow x = 1, y = 0 \Rightarrow (1,0) \rightarrow (1,0)$

S_2: $u=1$, $0 \le v \le 1 \Rightarrow x = u^2 - v^2 = 1 - v^2$, $y = 2\underset{1}{u}v = 2v$

$x = 1 - v^2$, $y = 2v \Rightarrow v = \frac{y}{2}$

$x = 1 - \left(\frac{y}{2}\right)^2$ or $x = 1 - \frac{y^2}{4}$: parabola



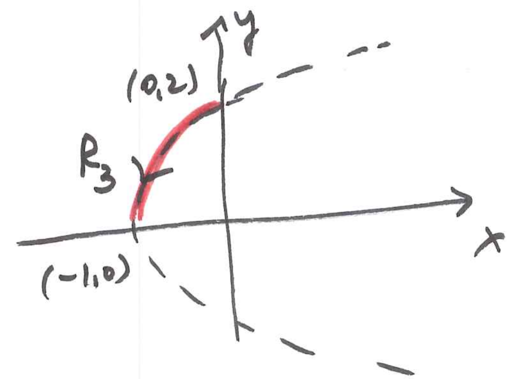
$u=1,\ v=0 \Rightarrow x = 1 - 0^2 = 1,\ y = 2v = 0 \Rightarrow (1,0)$

$u=1,\ v=1 \Rightarrow x = 1 - 1 = 0,\ y = 2 \cdot 1 = 2 \Rightarrow (0,2)$

S_3: $v=1$, $0 \le u \le 1 \Rightarrow x = u^2 - \overset{1}{v^2} = u^2 - 1$

$y = 2u\underset{1}{v} = 2u \Rightarrow u = \frac{y}{2}$

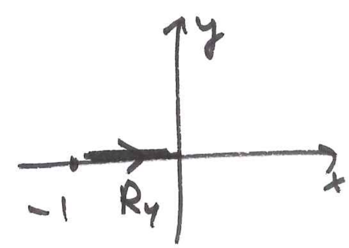
$\therefore x = u^2 - 1 = \left(\frac{y}{2}\right)^2 - 1$: parabola



S_4: $u=0$, $0 \le v \le 1 \Rightarrow x = \cancel{u^2}^{\,0} - v^2 = -v^2$

$y = 2\underset{0}{u}v = 0$

$0 \le v \le 1 \Rightarrow -1 \le x \le 0$



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To find the image of interior of S under transformation T, we find an image of any point from interior of S.

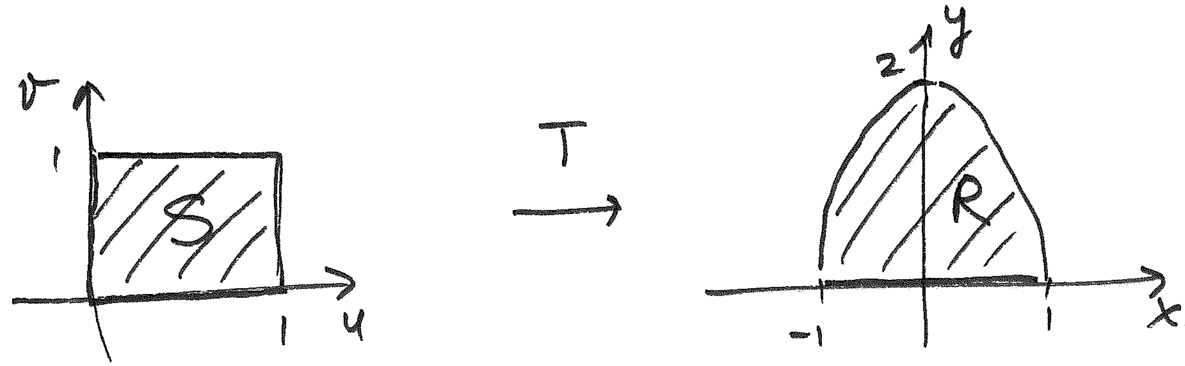
Let $u = \frac{1}{2}$, $v = \frac{1}{2}$

Then $x = u^2 - v^2 = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$

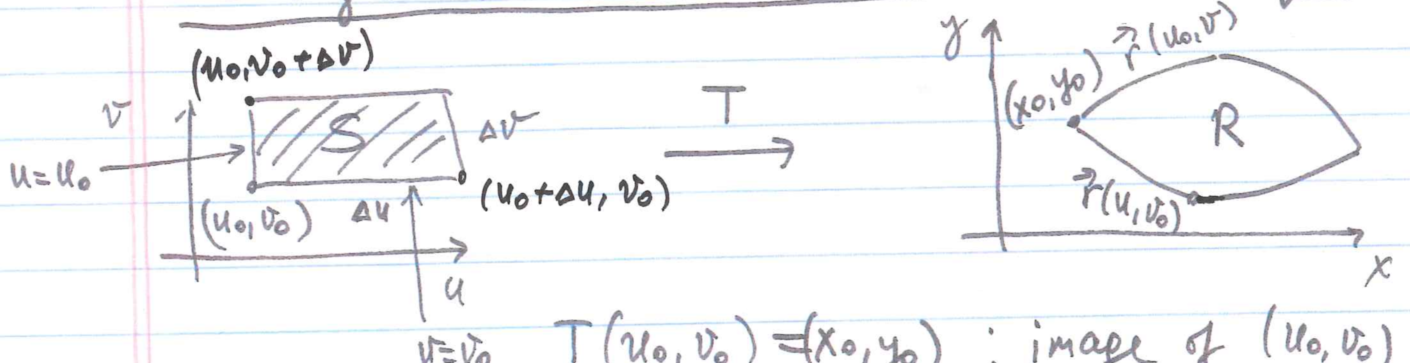
$y = 2uv = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \xrightarrow{T} \left(0, \frac{1}{2}\right)$ inside R

$\Rightarrow$ interior of S is transformed into interior of R



Change of variables in double integrals

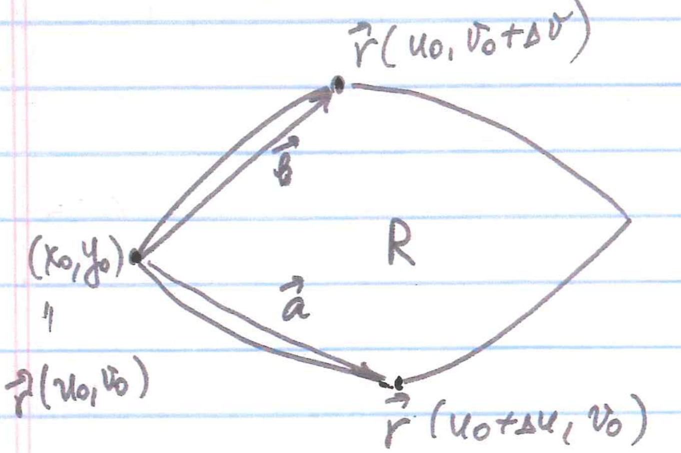


$T(u_0, v_0) = (x_0, y_0)$: image of (u_0, v_0)

$\vec{r}(u,v)$: position vector of image (u,v) point

$$\vec{r}(u,v) = \underbrace{g(u,v)}_{x}\,\vec{i} + \underbrace{h(u,v)}_{y}\,\vec{j}$$

$$T: \quad x = g(u,v)$$
$$\quad\;\; y = h(u,v)$$



$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$$
$$\approx \vec{r}_u(u_0, v_0)\cdot \Delta u$$

$$\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0)$$
$$\approx \vec{r}_v(u_0, v_0)\cdot \Delta v$$

Area of R $\approx$ Area of parallelogram determined by $\vec{a}, \vec{b}$ $= |\vec{a}\times\vec{b}| =$

$$= \left| \vec{r}_u(u_0,v_0)\cdot\Delta u \times \vec{r}_v(u_0,v_0)\Delta v \right| =$$

$$= \left| \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) \right| \Delta u\,\Delta v \quad =$$

$$\vec{r}(u,v) = g(u,v)\,\vec{i} + h(u,v)\,\vec{j}$$
$$\vec{r}_u = g_u(u,v)\,\vec{i} + h_u(u,v)\,\vec{j}$$

$$\vec{r}_v = g_v(u,v)\,\hat{\imath} + h_v(u,v)\,\hat{\jmath}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ g_u & h_u & 0 \\ g_v & h_v & 0 \end{vmatrix} = \hat{k} \begin{vmatrix} g_u & h_u \\ g_v & h_v \end{vmatrix} =$$

$g(u,v) = x, \quad h(u,v) = y$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \hat{k}$$

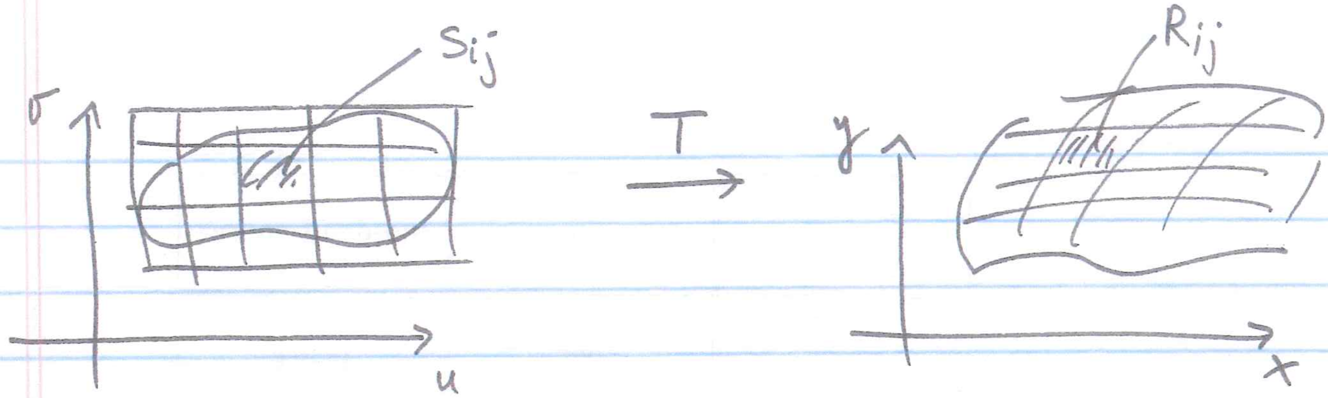
$$|\vec{a} \times \vec{b}| = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|} \cdot \Delta u \, \Delta v$$

Jacobian of transformation

$T: \quad x = g(u,v), \quad y = h(u,v)$

$\Rightarrow$ area of $R \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot \Delta u \, \Delta v$

abs. value



$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta A \approx$$

$$\approx \sum_{i=1}^{n} \sum_{j=1}^{m} f\big(g(u_i,v_j),\, h(u_i,v_j)\big) \overbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Bigg|_{(u_i,v_j)} \Delta u\, \Delta v}^{\Delta A}$$

$$\xrightarrow[n,m\to\infty]{} \iint_S f\big(g(u,v),\, h(u,v)\big) \cdot \underbrace{\left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv}_{dA}$$

__Def__ __Jacobian__ of transformation T given by

$x = g(u,v), \quad y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Hence,

$$\boxed{\begin{gathered}\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v))\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv \\ T:\ x=g(u,v),\quad y=h(u,v)\end{gathered}}$$

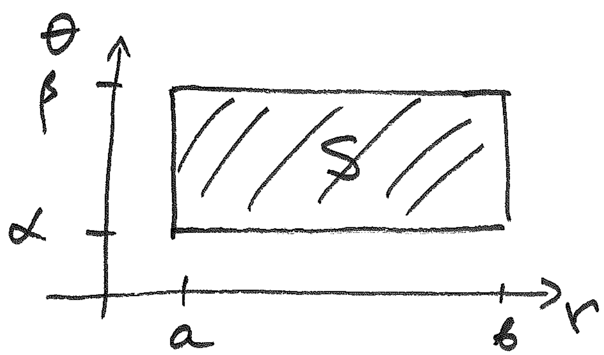
Note Compare this with

$$\int_a^b f(x)\,dx = \left|x=g(u)\right| = \int_c^d f(g(u))\,\underbrace{\frac{dx}{du}}_{g'}\,du$$

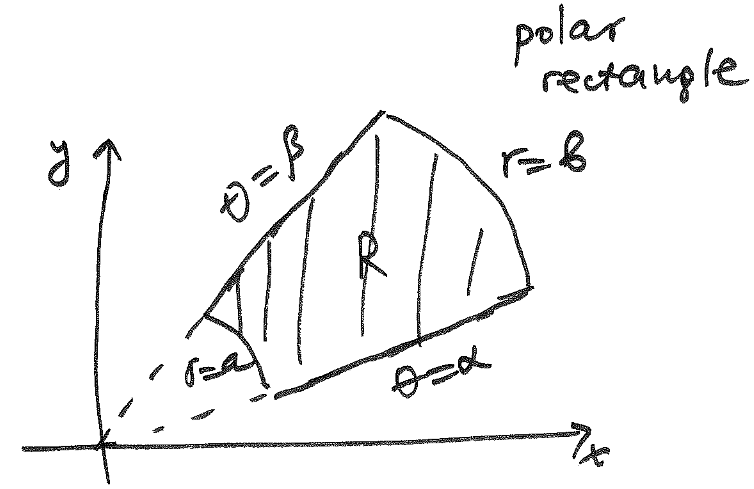
Ex Polar coordinates

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

T: $\quad g(r,\theta) \qquad h(r,\theta)$



$\xrightarrow{T}$



$S = \{(r,\theta):\ a \le r \le b,\ \alpha \le \theta \le \beta\}$

Note domain S in $r\theta$-plane (a rectangle) is much easier than R in xy-plane

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = $$

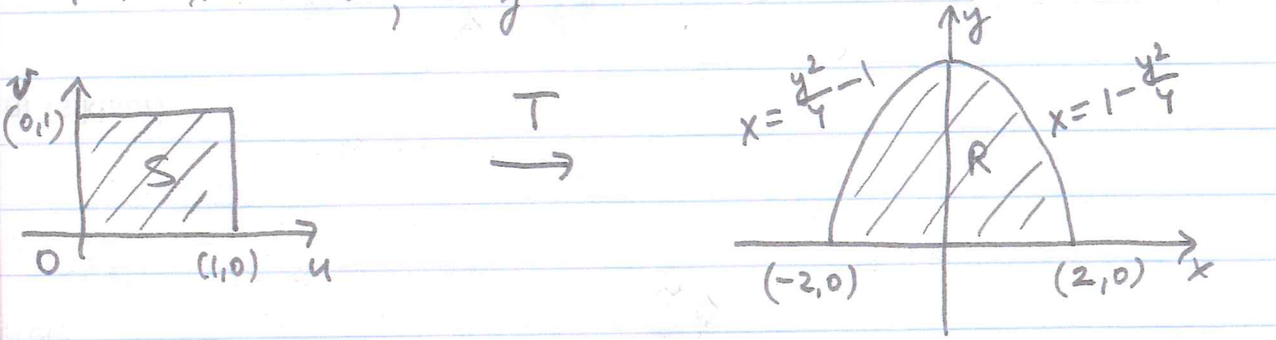
$$= r \ge 0$$

$$\therefore \iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta) \underbrace{\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|}_{r} dr\, d\theta =$$

$r \geq 0 \Rightarrow |r| = r$

$$= \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex $T: \; x = u^2 - v^2, \quad y = 2uv$



Find $\displaystyle\iint_R y \, dA$

Region R is bounded by parabolas $x = 1 - \frac{y^2}{4}$, $x = \frac{y^2}{4} - 1$ and x-axis, $y \geq 0$.

$$\iint_R y \underbrace{dA}_{= dx\,dy} = \iint_S \underbrace{2uv}_{y} \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv \;=$$

$$\underbrace{\frac{\partial(x,y)}{\partial(u,v)}}_{\text{Jacobian}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2) \geq 0$$

$$= \int_{u=0}^{1} \int_{v=0}^{1} \underbrace{2uv}_{y} \cdot \underbrace{4(u^2+v^2)}_{|J|} \, dv\,du = \ldots = 2$$

Triple Integrals

T : S in uvw-space $\longrightarrow$ R in xyz-space

T: $x = g(u,v,w)$, $y = h(u,v,w)$, $z = k(u,v,w)$

$$\text{Jacobian of } T = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\boxed{\iiint_R f(x,y,z)\,\underset{dx\,dy\,dz}{\underset{\shortparallel}{dV}} = \iiint_S f\big(g(u,v,w), h(u,v,w), k(u,v,w)\big) * \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$