Lecture 3

Notation:

V_2 : set of all 2D vectors $\langle a_1, a_2 \rangle$

V_3 : set of all 3D vectors $\langle a_1, a_2, a_3 \rangle$

. . . — — — — — -

V_n : set of all n-dimensional vectors $\langle a_1, a_2, \ldots, a_n \rangle$

12.3 Dot Product

Def Dot product / scalar product / inner product of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is the number/scalar

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{(in 3D)}$$

In 2D: $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle \Rightarrow \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

Ex Find

$$(\hat{\imath}+2\hat{\jmath}-3\hat{k})\cdot\left(-\tfrac{1}{2}\hat{\imath}+\hat{\jmath}\right) = \langle 1, 2, -3\rangle \cdot \langle -\tfrac{1}{2}, 1, 0\rangle =$$

$$= 1\cdot\left(-\tfrac{1}{2}\right) + 2\cdot 1 + (-3)\cdot 0 = -\tfrac{1}{2} + 2 = \tfrac{3}{2}$$

Properties of Dot Product

If $\vec{a}, \vec{b}, \vec{c}$ are from V_3, d, k are scalars, then

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
2. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
4. $(d\vec{a})\cdot\vec{b} = d(\vec{a}\cdot\vec{b})$
5. $\vec{a}\cdot\vec{0} = 0$

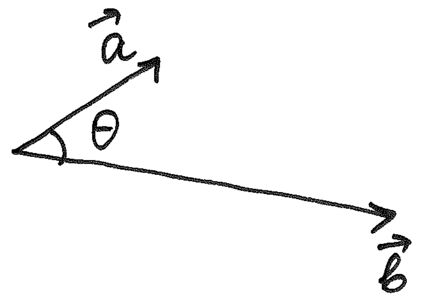
Proof of 2).

$$\vec{a}\cdot\vec{a} = \langle a_1, a_2, a_3\rangle \cdot \langle a_1, a_2, a_3\rangle =$$

$$= a_1\cdot a_1 + a_2\cdot a_2 + a_3\cdot a_3 =$$

$$= a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$$

Geometric Interpretation of Dot Product



Vectors $\vec{a}$ and $\vec{b}$ have the same starting point

θ: angle between $\vec{a}$ and $\vec{b}$

$0 \le \theta \le \pi$

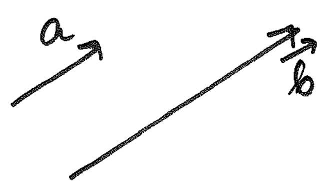
Then

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta}$$

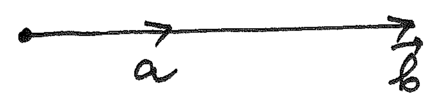
Proof uses the Law of Cosines.

Note

$\vec{a} \parallel \vec{b}$ and have the same direction



$\Rightarrow$



$\theta = 0 \Rightarrow \cos 0 = 1 > 0$

$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot 1 > 0$

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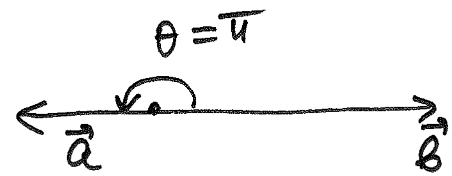
$\vec{a} \parallel \vec{b}$

but have

opposite direction



$\cos \pi = -1$

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot (-1) < 0$

<u>Def</u> Two vectors $\vec{a}$ and $\vec{b}$ are <u>orthogonal</u> / <u>perpendicular</u> if angle θ between them is $\theta = \frac{\pi}{2}$.

$\cos \frac{\pi}{2} = 0 \Rightarrow \vec{a} \cdot \vec{b} = 0$

<u>Notation</u>: $\vec{a} \perp \vec{b}$

<u>Claim</u>

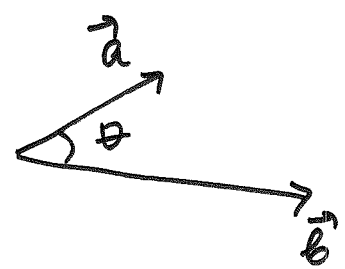
$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$

iff

<u>Corollary</u>: if $\vec{a}, \vec{b} \neq \vec{0}$, then we can find angle θ between them:

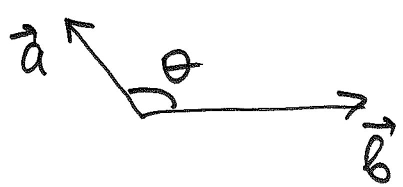
$$\boxed{\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}}$$

Note



We say that $\vec{a}$ and $\vec{b}$ have generally the same direction

$0 < \theta < \frac{\pi}{2} \Rightarrow \cos\theta > 0 \Rightarrow \vec{a}\cdot\vec{b} > 0$



we say: $\vec{a}$ and $\vec{b}$ have generally opposite direction

$\frac{\pi}{2} < \theta < \pi \Rightarrow \cos\theta < 0 \Rightarrow \vec{a}\cdot\vec{b} < 0$

Ex Find angle between $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$.

Solution

let θ: angle between $\vec{a}$ and $\vec{b}$

$\vec{a}\cdot\vec{b} = 2\cdot 3 + (-1)\cdot 2 + 1\cdot(-1) = 3$

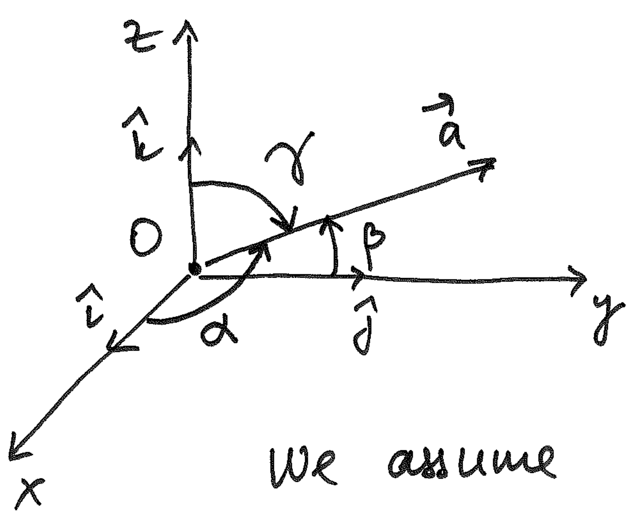
$|\vec{a}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$; $\quad |\vec{b}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$

$\therefore \quad \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \dfrac{3}{\sqrt{6}\cdot\sqrt{14}} = \dfrac{3}{\sqrt{84}} > 0 \Rightarrow \theta \in \text{I quadrant}$

or $\quad 0 < \theta < \frac{\pi}{2}$

$\Rightarrow \theta = \arccos \dfrac{3}{\sqrt{84}}$

Direction Angles and Direction Cosines



$\vec{a} = \langle a_1, a_2, a_3 \rangle$

Direction angles α, β, γ: angles that $\vec{a}$ forms with positive x-, y-, z-axes.

We assume $\quad 0 \le \alpha, \beta, \gamma \le \pi$

Def Direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are cosines of direction angles α, β, γ.

Recall

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|.|\vec{b}|}$$

To compute $\cos\alpha$, we take $\vec{b} = \hat{i}$ (think of α as the angle between $\vec{a}$ and $\hat{i}$):

$$\cos\alpha = \frac{\vec{a}\cdot\hat{i}}{|\vec{a}|.\underbrace{|\hat{i}|}_{1}} = \frac{\langle a_1, a_2, a_3\rangle . \langle 1, 0, 0\rangle}{|\vec{a}|} = \frac{a_1}{|\vec{a}|} \Rightarrow a_1 = |\vec{a}|\cdot\cos\alpha$$

Similarly,

$$\cos\beta = \frac{\vec{a}\cdot\hat{j}}{|\vec{a}|.\underbrace{|\hat{j}|}_{1}} = \frac{a_2}{|\vec{a}|}, \qquad \cos\gamma = \frac{\vec{a}.\hat{k}}{|\vec{a}|.|\hat{k}|} = \frac{a_3}{|\vec{a}|}$$

$$a_2 = |\vec{a}|\cos\beta \qquad a_3 = |\vec{a}|\cos\gamma$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2}{|\vec{a}|^2} + \frac{a_2^2}{|\vec{a}|^2} + \frac{a_3^2}{|\vec{a}|^2} = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} =$$

$$= \frac{|\vec{a}|^2}{|\vec{a}|^2} = 1$$

$$\therefore \quad \boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

We also showed:

$$a_1 = |\vec{a}|\cos\alpha, \quad a_2 = |\vec{a}|\cos\beta, \quad a_3 = |\vec{a}|\cos\gamma$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma \rangle = |\vec{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$$\Rightarrow \boxed{\frac{\vec{a}}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle}$$

unit vector in the direction of $\vec{a}$

Ex Find direction cosines and direction angles of

$$\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$$

Solution

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

$$\frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}\langle 2, -1, 2 \rangle = \langle \underbrace{\tfrac{2}{3}}_{\cos\alpha}, \underbrace{-\tfrac{1}{3}}_{\cos\beta}, \underbrace{\tfrac{2}{3}}_{\cos\gamma} \rangle = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

$\cos\alpha = \frac{2}{3} > 0 \Rightarrow \alpha \in \mathrm{I}$ quadrant $\Rightarrow \alpha = \arccos\frac{2}{3}$

$\cos\beta = -\frac{1}{3} < 0 \Rightarrow \beta \in \mathrm{II}$ quadrant $\Rightarrow \beta = \pi - \arccos\frac{1}{3}$

$\cos\gamma = \frac{2}{3} > 0 \Rightarrow \gamma = \arccos\frac{2}{3}$

Note

$$\cos(x-y) = \cos x \cdot \cos y + \sin x \cdot \sin y$$

$$\cos(\pi - y) = \underbrace{\cos\pi}_{-1} \cdot \cos y + \cancel{\sin\pi}^{\,0} \cdot \sin y = -\cos y$$

$$\Rightarrow \cos(\pi - y) = -\cos y \Rightarrow \pi - y = \arccos(-\cos y)$$

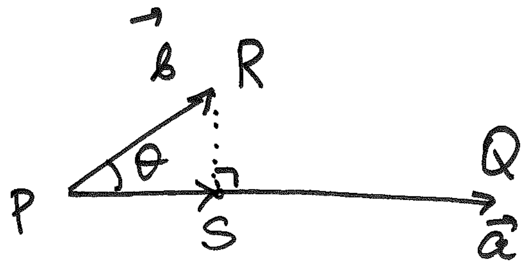
if $\cos\theta < 0$, i.e. $\frac{\pi}{2} < \theta < \pi$, we can compute θ using formular

$$\boxed{\theta = \pi - \arccos(-\cos\theta)}$$

In our example, $\cos\beta = -\frac{1}{3} < 0$

$$\Rightarrow \beta = \pi - \arccos\left(-\left(-\frac{1}{3}\right)\right) = \pi - \arccos\frac{1}{3}$$

Projections



$\vec{b} = \vec{PR}$, $\quad \vec{a} = \vec{PQ}$

$\vec{PS}$: vector projection of $\vec{b}$ onto $\vec{a}$, denoted by $\mathrm{proj}_{\vec{a}} \vec{b}$

θ : angle between $\vec{a}$ and $\vec{b}$, $\quad 0 \le \theta \le \pi$