Lecture 40

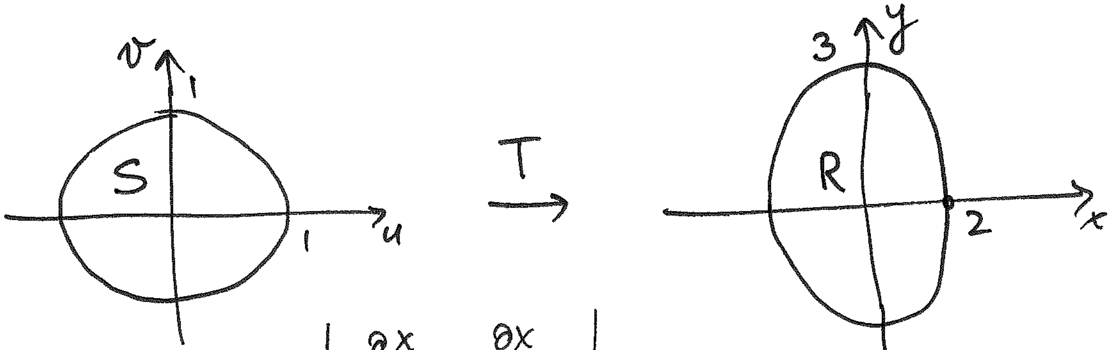
#17
S15.9

Use transformation $x=2u, y=3v$ to evaluate $\iint_R x^2 dA$, where R is the region bounded by the ellipse
$9x^2+4y^2=36$

Boundary $\xrightarrow{T}$ boundary

$$9x^2+4y^2=36 \Rightarrow \left| \begin{matrix} x=2u \\ y=3v \end{matrix} \right| \qquad 9\cdot(2u)^2+4\cdot(3v)^2=36 \qquad 9\cdot 4u^2+4\cdot 9v^2=36 \qquad \Rightarrow u^2+v^2=1$$

: unit circle w/ center at (0,0)



$$9x^2+4y^2=36 \; \Big| \; \frac{1}{36}$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

$$\boxed{dA=|J|\,du\,dv}$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}=6 \qquad |J|=6$$

$$\iint_R x^2 dA = \iint_S (2u)^2\cdot \underset{|J|}{6}\, du\,dv = 24\iint_{u^2+v^2\le 1} u^2\, du\,dv \overset{\text{polar coord.}}{=} 24\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\cos^2\theta\, r\,dr\,d\theta$$

$u=r\cos\theta$
$v=r\sin\theta$

$$=24\int_{\theta=0}^{2\pi}\cos^2\theta\,d\theta\cdot\int_{r=0}^{1} r^3 dr = 24\int_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta\cdot\left.\frac{r^4}{4}\right|_0^1 = 24\cdot\frac{1}{2}\cdot 2\pi\cdot\frac{1}{4}=\boxed{6\pi}$$

Ex Evaluate the integral by making an appropriate change of variables.

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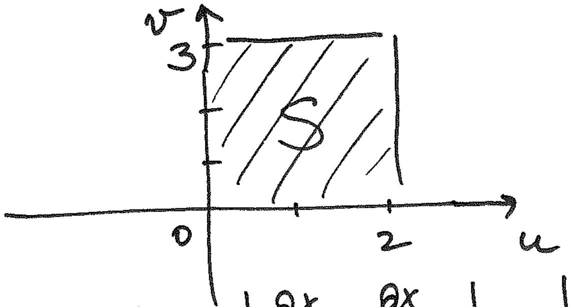
$$\iint_R (x+y)e^{x^2-y^2}\,dA \;\boxed{=}$$

where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

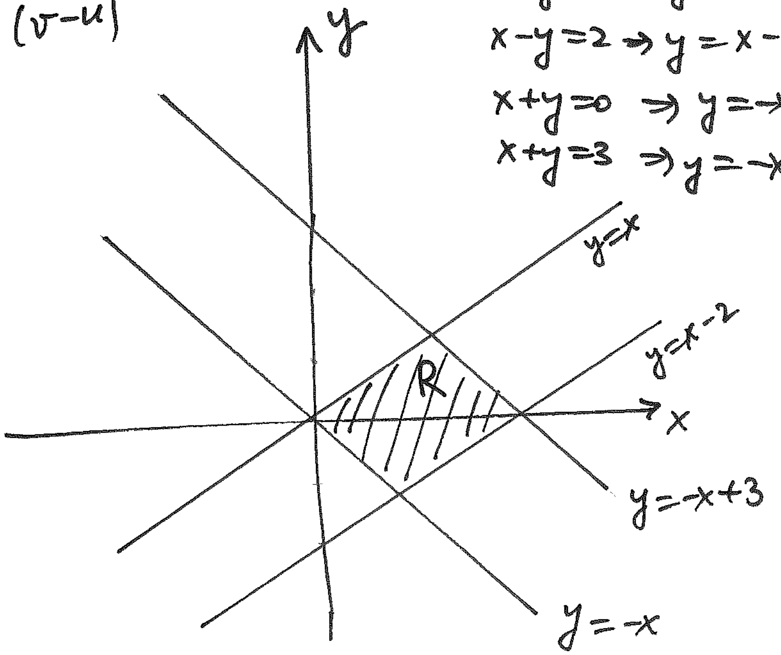
Let $u=x-y$, $v=x+y$: T $\Rightarrow$ $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(v-u)$: T^{-1}

$x-y=0 \Rightarrow y=x$
$x-y=2 \Rightarrow y=x-2$
$x+y=0 \Rightarrow y=-x$
$x+y=3 \Rightarrow y=-x+3$

Then $0\le u\le 2$, $0\le v\le 3$



$\xrightarrow{T}$



$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\boxed{=}\int_{v=0}^{3}\int_{u=0}^{2}\underset{e^z}{v e^{u\cdot v}}\cdot\underset{\uparrow\,|J|}{\frac{1}{2}}\,\overset{=dz}{du\,dv}=\left|\begin{array}{l} z=uv\\ dz=v\,du\\ u=0\Rightarrow z=0\\ u=2\Rightarrow z=2v\end{array}\right|=\frac{1}{2}\int_{v=0}^{3}\int_{z=0}^{2v}e^{z}\,dz\,dv=\frac{1}{2}\int_{v=0}^{3}e^{z}\Big|_{z=0}^{2v}dv=$$

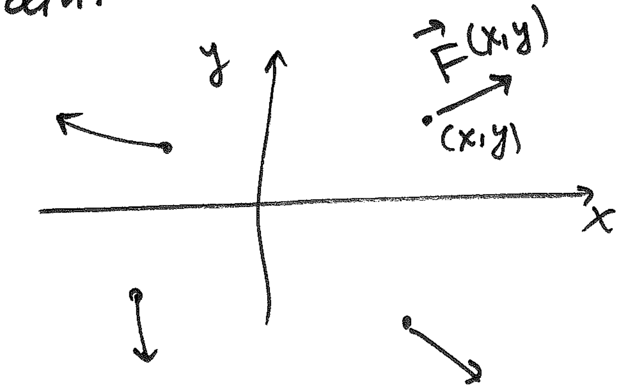
$$=\frac{1}{2}\int_{0}^{3}(e^{2v}-1)\,dv=\frac{1}{2}\left(\frac{1}{2}e^{2v}-v\right)\Big|_0^3=$$

$$=\frac{1}{2}\left[\left(\frac{1}{2}e^{6}-3\right)-\frac{1}{2}\right]=\frac{1}{2}\left[\frac{1}{2}(e^{6}-1)-3\right]$$

16.1 Vector Fields

Examples of vector fields: velocity field (wind, fluid), gravitational fields, force fields etc.

Vector field is a collection of vectors at every point in a domain.



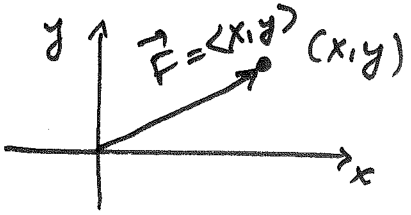
Def Let D be a domain in $\mathbb{R}^2$. Vector field $\vec{F}$ is a function that assigns every pt $(x,y) \in D$ a 2D vector $\vec{F}(x,y)$.

$$\underset{\text{vector function}}{\vec{F}(x,y)} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

scalar functions

P, Q: components of $\vec{F}$

Ex Position vector $\langle x, y\rangle$



Pt $(x, y) \iff$ position vector $\vec{x} = \langle x, y\rangle$

$$\Rightarrow \vec{F}(x, y) = \vec{F}(\vec{x}) = P\hat{\imath} + Q\hat{\jmath}$$

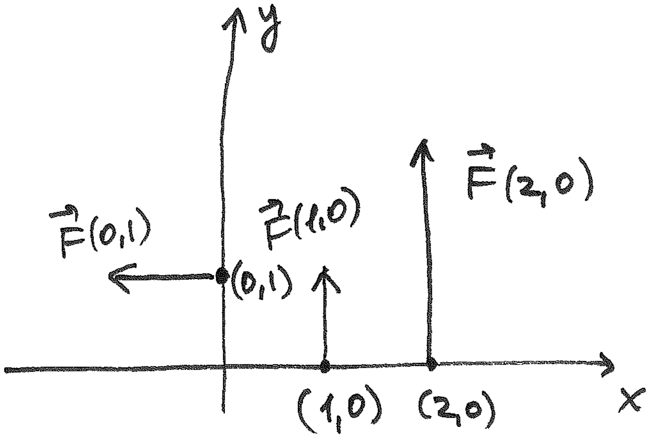
In 3D, def is similar: vector field $\vec{F}$ assigns a vector $\vec{F}(x, y, z) = \vec{F}(\vec{x})$

$$(x, y, z) \longleftrightarrow \langle x, y, z\rangle = \vec{x}$$

Ex Sketch vector field $\vec{F}(x, y) = -y\hat{\imath} + x\hat{\jmath}$

Solution:

$$\vec{F}(\underset{x}{\overset{\shortparallel}{1}}, \underset{y}{\overset{\shortparallel}{0}}) = \left(-y\hat{\imath} + x\hat{\jmath}\right)\Big|_{\substack{x=1 \\ y=0}} = 0\cdot\hat{\imath} + 1\cdot\hat{\jmath} = \langle 0, 1\rangle$$



$$\vec{F}(\underset{x}{\overset{\|}{2}},\underset{y}{\overset{\|}{0}}) = \left(-y\hat{i}+x\hat{j}\right)\Big|_{\substack{x=2\\y=0}} = 0\cdot\hat{i}+2\cdot\hat{j} = \langle 0,2\rangle$$

$$\vec{F}(\underset{x}{\overset{\|}{0}},\underset{y}{\overset{\|}{1}}) = \left(-y\hat{i}+x\hat{j}\right)\Big|_{\substack{x=0\\y=1}} = -\hat{i}+0\cdot\hat{j} = \langle -1,0\rangle$$

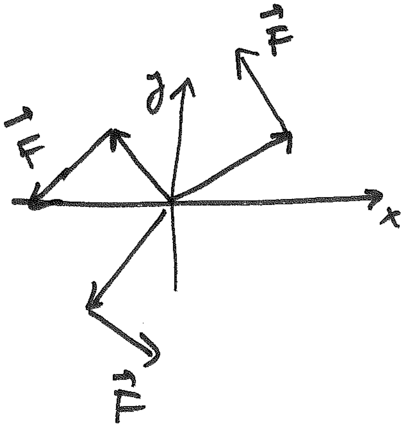
Note

$\vec{x} = \langle x, y\rangle$: position vector

Consider $\vec{F}(x,y) = -y\hat{i} + x\cdot\hat{j} = \langle -y, x\rangle$

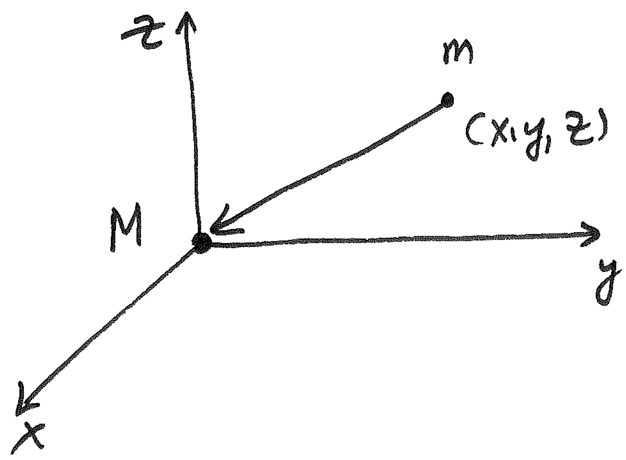
$\vec{x}\cdot\vec{F} = \langle x,y\rangle\cdot\langle -y,x\rangle = -xy+yx = 0 \Rightarrow \vec{x} \perp \vec{F}$

$\therefore$ $\vec{F}$ is tangent to the circle centered at origin with radius $|\vec{x}| = \sqrt{x^2+y^2}$.



Def If vector field $\vec{V}(x,y)$ represents velocity at every pt (x,y), we have velocity field.

Ex Force field: gravitational field



Gravitational field exerted by mass M on mass m at (x,y,z) is

$$\vec{F} = -\frac{GmM}{|\vec{x}|^3}\,\vec{x}$$

M, m : masses

G: gravitational const

$r = |\vec{x}|$: distance between M & m

$\frac{\vec{x}}{|\vec{x}|}$: unit vector

$$|\vec{F}| = \frac{GmM}{r^2}$$

$$\vec{F} = |\vec{F}| \cdot \left(-\frac{\vec{x}}{|\vec{x}|}\right) = -\frac{GmM}{|\vec{x}|^2} \cdot \frac{\vec{x}}{|\vec{x}|} = -\frac{GmM}{|\vec{x}|^3} \cdot \vec{x}$$

Ex Gravitational field

$$\vec{F} = -\frac{GmM}{|\vec{x}|^3}\vec{x}$$

Gravitational potential $f(x,y,z) = \dfrac{GmM}{\sqrt{x^2+y^2+z^2}} = GmM\,(x^2+y^2+z^2)^{-1/2}$

Need to show $\nabla f = \vec{F}$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f_x = \frac{\partial f}{\partial x} = GmM\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2} \cdot 2x = -GmM\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$|\vec{x}| = \sqrt{x^2+y^2+z^2} \qquad = -GmM\frac{x}{|\vec{x}|^3}$$

Similarly, $f_y = -GmM\dfrac{y}{|\vec{x}|^3}$ $\qquad f_z = -GmM\dfrac{z}{|\vec{x}|^3}$

Ex Electric field

Q: electric charge at origin

According to Coulomb's law:

$$\vec{F}(\vec{x}) = \frac{\varepsilon Q q}{|\vec{x}|^3}\vec{x}$$: electric field force exerted by charge Q on charge q located at (x, y, z)

ε: Coulomb's const

$Qq > 0$ (Q, q of the same sign) : force is repulsive

$Qq < 0$ (Q, q of opposite sign): force is attractive

Force per unit charge:

$$\frac{\vec{F}(\vec{x})}{q} = \frac{\varepsilon Q}{|\vec{x}|^3}\vec{x}$$: electric field of Q

Gradient Fields

2D $f=f(x,y)$

$\nabla f(x,y) = f_x(x,y)\,\hat{i} + f_y(x,y)\,\hat{j}$: gradient vector field in 2D

3D $f=f(x,y,z)$

$\nabla f(x,y,z) = f_x(x,y,z)\,\hat{i} + f_y(x,y,z)\,\hat{j} + f_z(x,y,z)\,\vec{k}$: gradient vector field in 3D

Def Vector field $\vec{F}$ is conservative if there is a scalar function f : $\vec{F} = \nabla f$

f: potential function of $\vec{F}$

Ex velocity field, electric field, gravitational field are conservative

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle -GmM \frac{x}{|\vec{x}|^3}, -GmM \frac{y}{|\vec{x}|^3}, -GmM \frac{z}{|\vec{x}|^3} \rangle =$$

$$= -GmM \frac{1}{|\vec{x}|^3} \underbrace{\langle x, y, z)}_{\vec{x}} = -GmM \frac{\vec{x}}{|\vec{x}|^3} = \vec{F} \quad ✓$$

$\therefore$ $\vec{F}$ is conservative w/ potential function

$$f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$$