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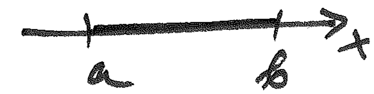
Lecture 41

16.2 Line Integrals

$\int_a^b f(x)\,dx$ $\quad$ $f(x)$ is given on $[a, b]$

integral over segment $[a, b]$



Generalize to integrate over a curve.

2D: $\quad C$: plane curve

$x = x(t)$, $\quad y = y(t)$, $\quad a \le t \le b$

or $\vec{r} = \vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

Assume curve C is smooth, i.e. $\vec{r}\,'$ is continuous and $\vec{r}\,' \ne 0$.

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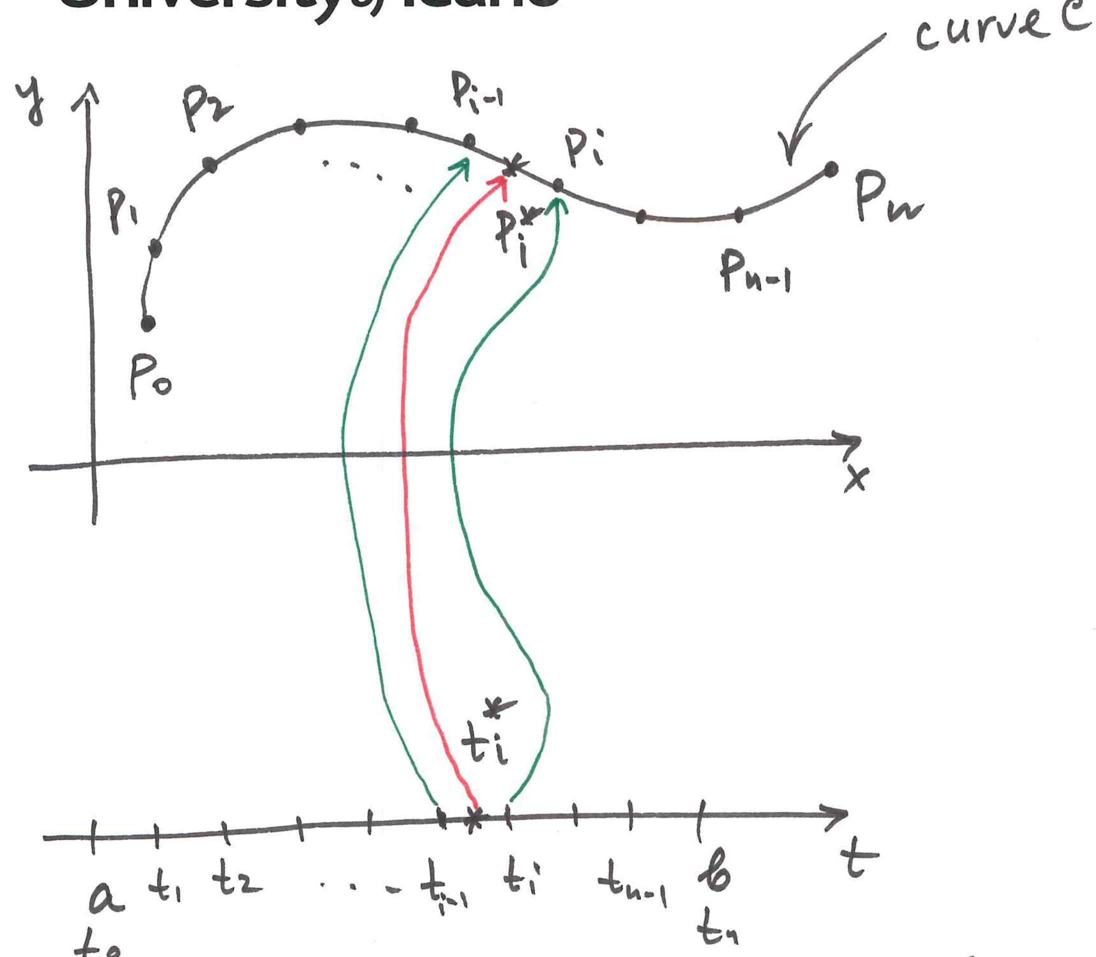


$P_0(x_0, y_0)$

$x_0 = x(t_0), \; y = y(t_0)$

etc.

Points $P_0, P_1, P_2, \ldots, P_n$ subdivide curve C into arcs.

Assume $f(x,y)$ has domain that includes curve C.

$P_i^*(x_i^*, y_i^*)$

$x_i^* = x(t_i^*), \; y_i^* = y(t_i^*)$

$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$

Form $\displaystyle\sum_{i=1}^{n} f(x_i^*, y_i^*) \, \Delta S_i$: looks like a Riemann sum

length of arc $P_{i-1} P_i$

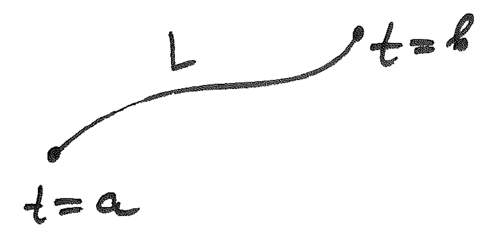
<u>Def</u> <u>line integral</u> of f along curve C <u>with respect to arclength</u> is

$$\int_C f(x,y)\,ds \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

if limit exists.

Recall

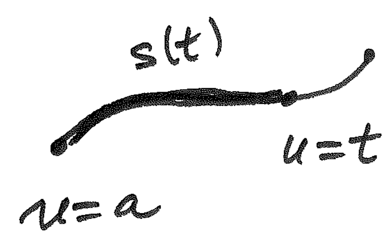
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$



length of curve C

Arclength function

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\,du$$



length of a part of C from $u=a$ to $u=t$

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$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{or} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

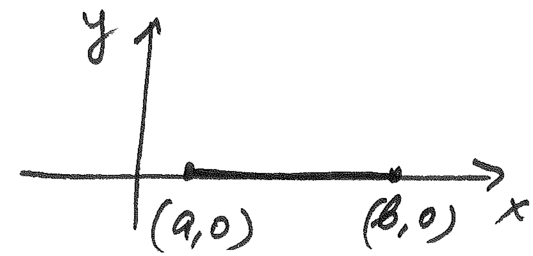
Then we can write

$$\boxed{\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

line $\int$ wrt arc-length

Partial case: curve C is line segment from $(a,0)$ to $(b,0)$

Segment: $x=x,\ y=0 \qquad a \le x \le b$



Then

$$\int_C f(x,y)\, ds = \int_a^b f(x,0) \sqrt{\underbrace{\left(\frac{dx}{dx}\right)^2}_{=1} + \underbrace{\left(\frac{dy}{dx}\right)^2}_{=0}}\, dx =$$

$$= \int_a^b f(x,0)\, dx \ : \text{ usual } \int \text{ on } [a,b]$$

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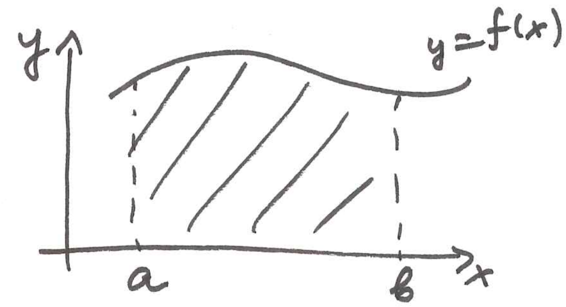
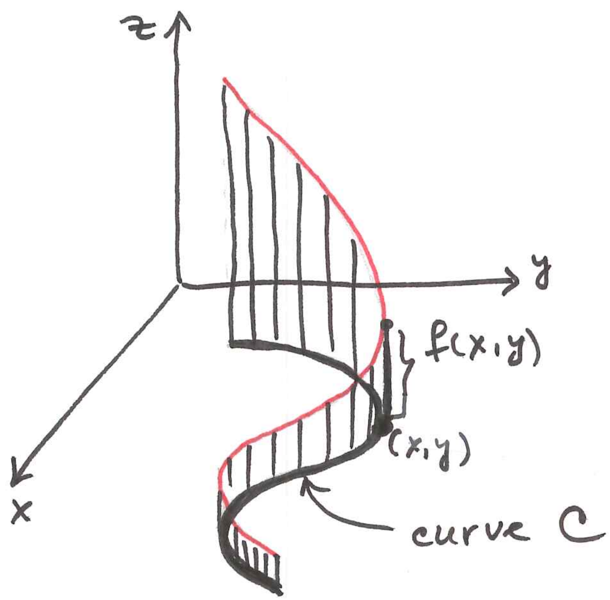
$\int_a^b f(x)\,dx$ $\quad f(x) \geq 0$

is area under curve $y = f(x)$ and above x-axis, $a \leq x \leq b$



Let $f(x,y) \geq 0$. What is geometric meaning of $\int_C f(x,y)\,ds$?



$\int_C f(x,y)\,ds$ represents area of one side of "curtain", whose base (projection on xy-plane) is curve C and height is given by $f(x,y)$ at every pt $(x,y) \in C$.

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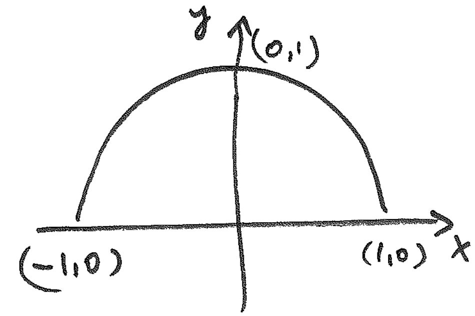
Ex Evaluate $\int_C (2+x^2y)\,ds =$



C: upper half of circle $x^2+y^2=1$

$x=\cos t,\quad y=\sin t \qquad 0\le t\le \pi$

$$= \int_{t=0}^{\pi} (2+\cos^2 t\cdot\sin t)\sqrt{\underbrace{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}_{(-\sin t)^2+(\cos t)^2\,=\,1}}\,dt = \int_0^{\pi}(2+\cos^2 t\cdot\sin t)\,dt =$$

$$= \int_0^{\pi} 2\,dt + \int_0^{\pi}\cos^2 t\cdot\sin t\,dt = \left|\begin{array}{l} u=\cos t \\ du=-\sin t\,dt \\ t=0 \Rightarrow u=1 \\ t=\pi \Rightarrow u=-1 \end{array}\right| = 2\pi + \underbrace{\int_{-1}^{1} u^2\,du}_{\text{even } f^n} = 2\pi + 2\int_0^1 u^2\,du =$$

$$= 2\pi + \frac{2}{3}u^3\Big|_0^1 = \boxed{2\pi + \frac{2}{3}}$$