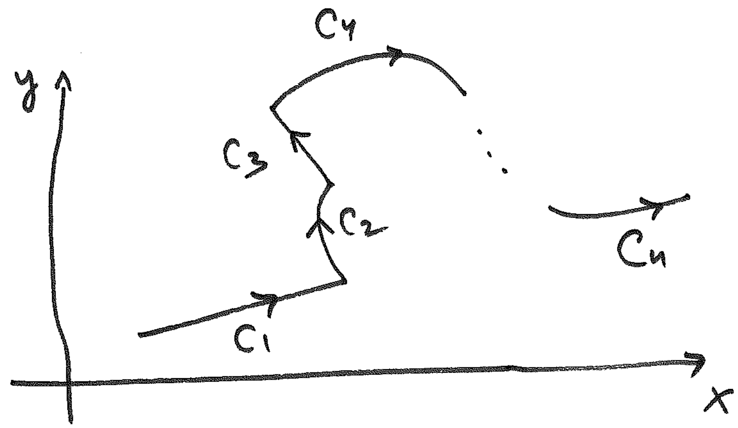
Lecture 42

Line Integrals (Cont'd)



C: piecewise smooth curve

$C = C_1 \cup C_2 \cup \ldots \cup C_n$

Then

$$\int_{C = C_1 \cup C_2 \cup \ldots \cup C_n} f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds$$

Note: each segment C_i may be parametrized using different formulas.

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Other types of line integrals

$f(x,y)$, replace ΔS_i with $\Delta x_i = x_i - x_{i-1}$

or with $\Delta y_i = y_i - y_{i-1}$

Form Riemann like sums. Take a limit as $n \to \infty$ to get

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

line integrals of $f(x,y)$ along curve C wrt x and y, respectively

Recall

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta S_i$$

: line integral wrt arc length

Curve C: $\quad x = x(t), \quad y = y(t) \quad a \le t \le b$

$dx = x'(t)\,dt \qquad dy = y'(t)\,dt \qquad ds = \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

$\therefore \displaystyle\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))\, x'(t)\,dt$: line $\int$ wrt x

$\displaystyle\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\, y'(t)\,dt$: — " — y

$\displaystyle\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$: — " — wrt arclength

Note: Line integrals wrt x and y may occur together (along the same curve C) $\Rightarrow$ we write

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

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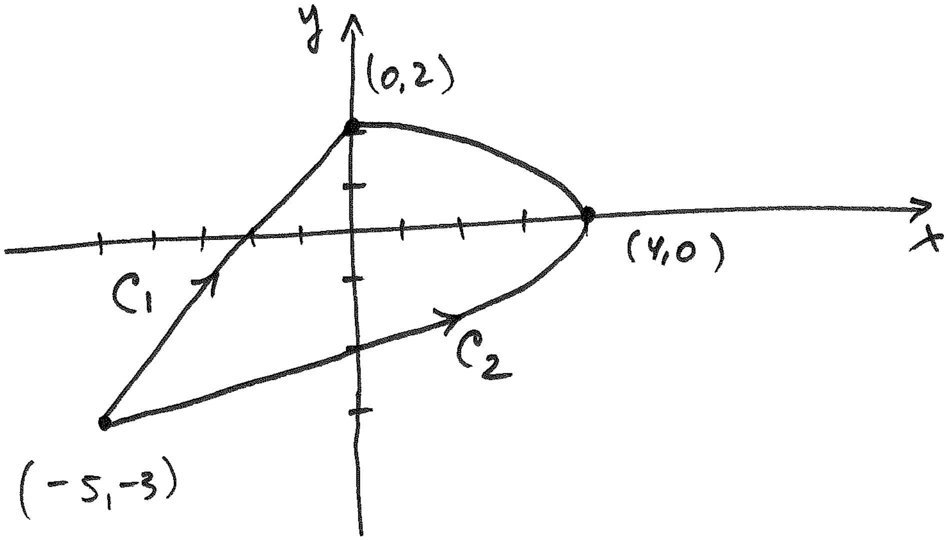
Ex Evaluate $\displaystyle\int_C y^2\,dx + x\,dy$

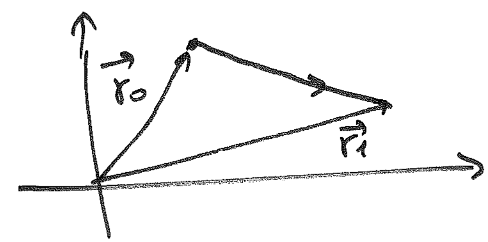
where

(a) $C = C_1$: line segment from $(-5,-3)$ to $(0,2)$

(b) $C = C_2$: arc of parabola from $(-5,-3)$ to $(0,2)$



Recall



$\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1, \quad 0 \le t \le 1$

line segment from tip of $\vec{r}_0$ to tip of $\vec{r}_1$

Here, $\vec{r}_0 = \langle -5,-3\rangle \qquad \vec{r}_1 = \langle 0,2\rangle$

$\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1 = (1-t)\langle -5,-3\rangle + t\langle 0,2\rangle = \langle -5(1-t), -3(1-t)\rangle +$

$+ \langle 0, 2t\rangle = \langle -5+5t+0, \; -3+3t+2t\rangle = \langle \underbrace{5t-5}_{x(t)}, \; \underbrace{5t-3}_{y(t)} \rangle$

$x(t) = 5t - 5$ $\quad$ $y(t) = 5t - 3$ $\quad$ $0 \le t \le 1$

$x'(t) = 5$ $\quad$ $y'(t) = 5$

Hence,

$$\int_{C_1} y^2 \underbrace{dx}_{x'(t)\,dt} + x \underbrace{dy}_{y'(t)\,dt} = \int_0^1 (5t-3)^2 \cdot 5\,dt + (5t-5)\cdot 5\,dt =$$

$$= 5\int_0^1 \left((5t-3)^2 + 5t - 5\right) dt = \ldots - \frac{5}{6}$$

(b) $C = C_2$: $\quad x = 4 - y^2$ $\quad -3 \le y \le 2$ $\quad \frac{dx}{dy} = -2y$

i.e. $x = 4 - y^2, \quad y = y, \quad -3 \le y \le 2$ $\quad \frac{dy}{dy} = 1$

or $x = 4 - t^2, \quad y = t, \quad -3 \le t \le 2$

$$\int_{C_2} y^2 \underbrace{dx}_{= x'\,dy} + x \underbrace{dy}_{= 1 \cdot dy} = \int_{-3}^{2} y^2 \cdot (-2y)\,dy + (4 - y^2) \cdot 1\,dy =$$

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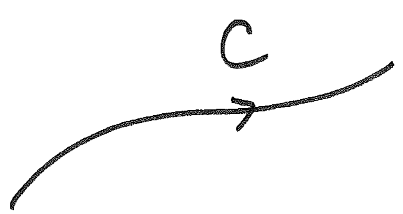
$$= \int_{-3}^{2} \left(-2y^3 + 4 - y^2 \right) dy = \ldots = 40\tfrac{5}{6}$$

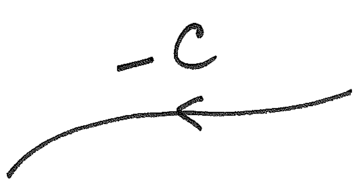
Note line integrals depend not only on endpoints but also on a path, i.e.

$$\int_{C_1} f \neq \int_{C_2} f \quad \text{in general}$$

Note



$$\int_{-C} f(x,y)\, dx = -\int_{C} f(x,y)\, dx$$



$$\int_{-C} f(x,y)\, dy = -\int_{C} f(x,y)\, dy$$

but

$$\int_{-C} f(x,y)\, dS \;=\; \int_{C} f(x,y)\, dy$$

wrt arclength

because $\Delta S > 0$