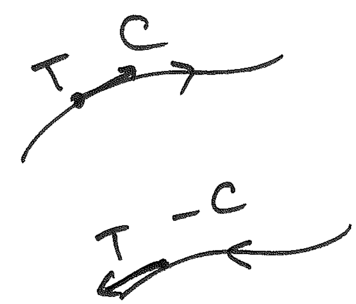
Lecture 43

<u>Note</u> even though

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$$

wrt arclength



$$\Rightarrow \int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$$

since unit tangent vector $\vec{T} \to -\vec{T}$ when $C \to -C$

<u>Note</u> Let vector field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b (P\hat{i}+Q\hat{j}+R\hat{k})\cdot(x'(t)\hat{i}+y'(t)\hat{j}+z'(t)\hat{k})\,dt$$

$$= \int_a^b \left[P(x(t),y(t),z(t))\cdot x'(t) + Q(\dots)\cdot y'(t) + R(\dots)\cdot z'(t)\right]dt =$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

$$\therefore \quad \boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz \quad \text{where } \vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}}$$

#21 S 16.2 Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is given by the vector function $\vec{r}(t)$.

$\vec{F}(x,y,z) = \sin x\cdot\hat{\imath} + \cos y\cdot\hat{\jmath} + xz\cdot\vec{k}$, $\quad \vec{r}(t) = \underbrace{t^3}_{x}\cdot\hat{\imath} - \underbrace{t^2}_{y}\cdot\hat{\jmath} + \underbrace{t}_{z}\vec{k}$, $0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \qquad \vec{r}\,'(t) = \langle 3t^2, -2t, 1\rangle$$

$$= \int_0^1 \langle \sin x(t), \cos y(t), x(t)\cdot z(t)\rangle \cdot \langle 3t^2, -2t, 1\rangle\, dt =$$

$$= \int_0^1 \langle \sin t^3, \underbrace{\cos(-t^2)}_{\cos t^2}, \underbrace{t^3\cdot t}_{t^4}\rangle \cdot \langle 3t^2, -2t, 1\rangle\, dt =$$

dot product

$$= \int_0^1 \left(\sin t^3\cdot 3t^2 - \cos t^2\cdot 2t + t^4\cdot 1\right) dt = \int_0^1 3t^2 \sin t^3\, dt - \int_0^1 2t\cdot\cos t^2\, dt +$$

$u = t^3$, $du = 3t^2 dt$ $\qquad u = t^2$, $du = 2t$

$$+ \int_0^1 t^4\, dt = \left(-\cos t^3 - \sin t^2 + \frac{t^5}{5}\right)\Big|_0^1 =$$

$$= \left(-\cos 1 - \sin 1 + \frac{1}{5}\right) - (-1) = \boxed{\frac{6}{5} - \cos 1 - \sin 1}$$

16.3 Fundamental Theorem for Line Integrals

Recall

$$\int_a^b \underset{\substack{\uparrow \\ \text{rate of change} \\ \text{of } F}}{F'(x)}\,dx = \underbrace{F(b) - F(a)}_{\substack{\text{net change} \\ \text{of } F}}$$

Fundamental Theorem of Calculus

Thm (Fundamental Thm for Line Integrals)

Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Let f differentiable and ∇f be continuous on C. Then

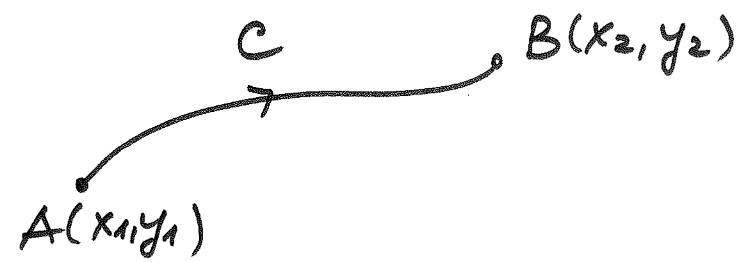
$$\int_C \nabla f \cdot d\vec{r} = \underbrace{f(\vec{r}(b)) - f(\vec{r}(a))}_{\text{net change of } f}$$

Note $f = f(x,y)$ or $f = f(x,y,z)$

University of Idaho

<u>Note</u> : ∇f is a conservative vector field

<u>Ex</u> $f = f(x,y)$



$$\int_C \nabla f \cdot d\vec{r} \underset{\substack{\text{Thm} \\ \text{for line} \\ \int}}{\overset{\text{fundam}}{=}} f(x_2, y_2) - f(x_1, y_1)$$

INDEPENDENCE OF PATH

In general,

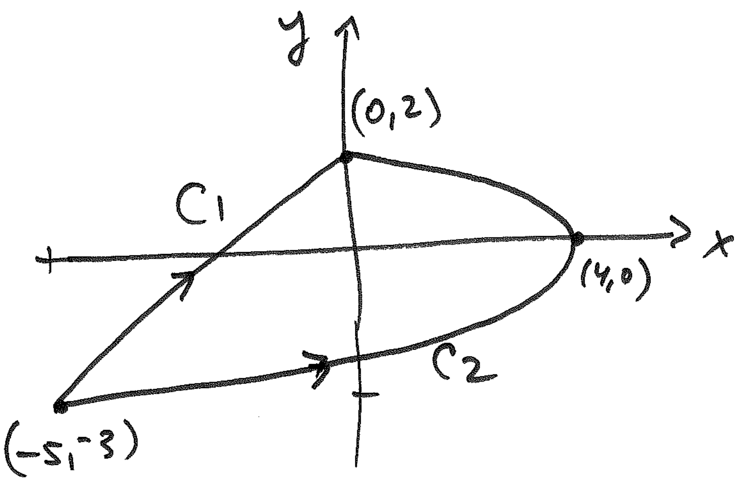
$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$$

even when initial and terminal points of C_1 and C_2 are the same.

University of Idaho

Ex

Recall



We saw last time (see Lecture 42)

$$\int_{C_1} y^2 dx + x\, dy \neq \int_{C_2} y^2 dx + x\, dy$$

From the fundamental theorem for line integrals,

when $\vec{F} = \nabla f$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} \overset{\text{fund. thm for line } \int}{=} f(\vec{r}(b)) - f(\vec{r}(a)) \overset{\text{fund. Thm for line } \int}{=}$$

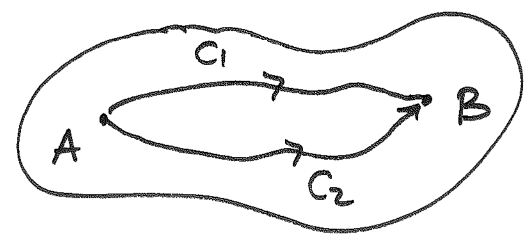
$$= \int_{C_2} \nabla f \cdot d\vec{r}$$

if C_1 and C_2 have the same initial and terminal points, i.e.

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$

for a conservative field ∇f.

Def Let $\vec{F}$ be a continuous vector field in some domain D. We say that $\int_C \vec{F} d\vec{r}$ is INDEPENDENT OF PATH if

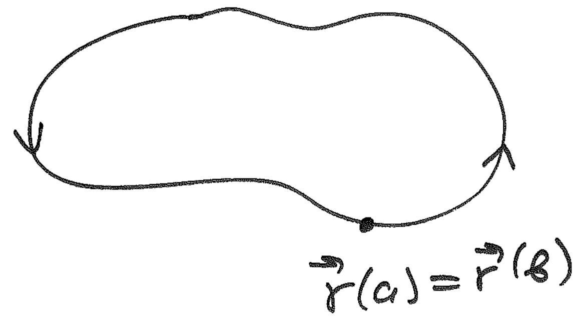


$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

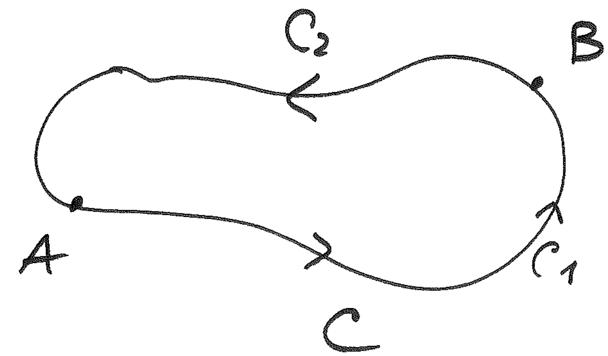
for any curves C_1, C_2 within domain D, that have the same initial and terminal points.

Note We saw that line integral of a conservative vector field is independent of path.

Def Closed curve: $\vec{r}(a) = \vec{r}(b)$, i.e. initial and terminal points coincide.



Let $\int_C \vec{F} \cdot d\vec{r}$ be independent of path, C be a closed curve.



$$\int_C \vec{F} d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} =$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{-C_2} \vec{F} \cdot d\vec{r} = 0$$ since $\int_C \vec{F} d\vec{r}$ is independent of path

We showed that if $\int_C \vec{F} d\vec{r}$ is independent of path, then

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

where C is a closed contour.

Notations for closed contour : $\int_C \vec{F}\cdot d\vec{r}$ or $\oint_C \vec{F}\cdot d\vec{r}$ or $\oint_C \vec{F}\cdot d\vec{r}$

<u>Thm</u> $\int_C \vec{F}\cdot d\vec{r}$ is <u>independent of path</u> iff

$\int_C \vec{F}\cdot d\vec{r} = 0$ for any <u>closed</u> curve C.

<u>Thm</u> The <u>only</u> vector field that are <u>independent of path</u> are <u>conservative</u> vector fields $\vec{F}$ (i.e. there exists a scalar function f: $\vec{F} = \nabla f$)
potential function

<u>Q</u> Assume that we know that vector field $\vec{F}$ is conservative, i.e. there exists a function f: $\vec{F} = \nabla f$.
How do we find f?

University of Idaho

$\vec{F} = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

$\vec{F} = \nabla f = \langle f_x, f_y \rangle = f_x\,\vec{i} + f_y\,\vec{j}$

$\therefore$ $\boxed{f_x = P(x,y), \qquad f_y = Q(x,y)}$

$\dfrac{\partial P}{\partial y} = f_{xy}$ $\qquad$ $\dfrac{\partial Q}{\partial x} = f_{yx}$

but $f_{xy} = f_{yx}$

$\therefore$ $\boxed{\text{if } \vec{F} \text{ is conservative} \Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$

Ex $\vec{F}(x,y) = (3 + 2xy)\,\vec{i} + (x^2 - 3y^2)\,\vec{j}$

Find f such that $\nabla f = \vec{F}$.

Solution:

$P = 3 + 2xy$ $\qquad$ $Q = x^2 - 3y^2$

$f_x = P(x,y) \Rightarrow f_x = 3 + 2xy$

Integrate wrt x:

$$f = 3x + x^2 y + \underbrace{g(y)}_{\text{arbitrary function of } y}$$

Then

$$Q = f_y = x^2 + g'(y) \Rightarrow \cancel{x^2} + g'(y) = \cancel{x^2} - 3y^2$$

$$\Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + \underbrace{K}_{\text{arbitrary const}}$$

$\therefore$ $\boxed{f(x,y) = 3x + x^2 y - y^3 + K}$ potential function